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RENTAL RATE AND THE DYNAMICS OF THE MARKET PRICE OF CAPITAL

Rental rate gives the opportunity cost of a machine, that is, it accounts for the opportunities forgone by using the machine or self-renting it instead of renting it out to someone else. While the traditional approach studies how the rate at which a machine can be rented depends on the market price of that machine, it is interesting to trace the time path of the price of capital in relation to a given expected rental rate. When the rental rate is relatively stable and firms do not expect it to change with time the intertemporal equilibrium market price of the machine is the initial price. When market participants expect the rental rate to increase, the price of the machine can increase or decrease exponentially depending on the initial price level. Given that rental rate is expected to fall, the market price of capital will grow exponentially.
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Rental rate is defined as the rate at which an asset can be rented for a given time period. Rental rate gives the opportunity cost of the asset, that is, it accounts for the opportunities forgone by using the asset or self-renting it instead of renting it out to someone else. Assets can be rented at the rental market and can take any form – from a machine to a patented technology to a trademark that can be used by others who appropriate the usufruct right from the owner without actually buying the asset[2]. The right of use is leased for a specified period of time in return for periodic payment of a stipulated price, the rent. Modern operating leasing involves the renting of industrial machinery, office equipment or even complete industrial installations. The rental of real and personal property is arranged through a lease contract, which obliges the lessor to grant the lessee the use of the rented object for a specified period of time, while the lessee is obliged to pay the lessor the stipulated rent. Leasing represents a modern type of business and a hybrid form of contractual arrangement together with joint ventures, franchising, factoring, know-how contracts, credit card contracts, etc.

I. Introduction

A typical form of rental relations for real estate property exists on the market for rented apartments. They are often the object of government regulation. Rent controls have largely been used in the big cities in the US in the past. In Germany a large percentage of the population lives in rented apartments, which have been

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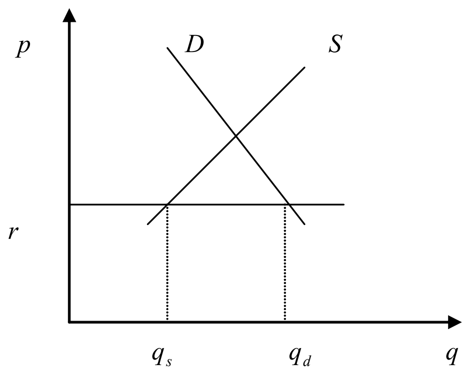
[2] Instead of a purchase price the lessee of a technology or a trade/brand name would pay royalties to its owner, the leaser.

strictly regulated by the state since World War I (Furubotn and Richter, 2000, 134). Controlling the rental market for real estate has resulted in significant deterioration of the quality and maintenance of buildings and has exacerbated the housing problem in various countries. Former socialist states have had such regulations for many years, too. That has partly been the result of abandoning any kind of market and the need for government "care" for the population. In some extreme cases people would not have the right to own their apartments but only to rent them.[3] In the early years of socialism Bulgarian citizens did not have the right to buy the real estate they rented. In later years they were restricted from owning more than one apartment. Most buildings were therefore owned by the state until the beginning of the transition.

Generally, the problem of rented apartments is presented in popular literature with a simple demand and supply curve. As the state regulates the prices of blocks of flats, there is undersupply of housing and impossibility for the rental market to clear. (See Figure 1)

Figure 1

Undersupply of apartments for rent in the conditions of regulation



It is easy to observe that at the regulated rent r, artificially maintained lower than the equilibrium rental rate by the government, the quantity demanded exceeds the quantity supplied. The effects of controlling rents and rationing apartments in Eastern Europe resulted in queuing and bribery. Rented apartments were shoddy and incentives to supply more of them were lacking. Inherent problems arose between tenants and owners, a relationship very well described by Furubotn and Richter (2000, 134). Occasionally, landlords would seek ways such as bundling to charge the tenants the full economic price of the rented apartment[4].

"A strong belief exists that tenants need state protection regarding the rents charged and that limitations must also be placed on the owner's right to give notice to vacate rented premises. In consequence, an owner's freedom to price his property and his right to control the terms of its use have been severely attenuated by government. The climate is such that market forces cannot fully perform their

[3] An example is the city of Moscow where the impossibility to buy apartments and the rental system have led to severe shortages of living space and multiple cases of several families living in a small flat.

[4] For example, renting the apartment while paying separately for expensive furniture or paying additionally for extra services provided by the landlord.

normal function in the rental housing market and, inter alia, considerable government support is necessary to stimulate the supply of apartments."

Liberating the rental market for apartments would lead to a return to the equilibrium rent and quantity of apartments. As the increase in rents was projected to be sudden and substantial, many governments in Eastern Europe postponed the full liberalization of rents until the mid-1990s. The gradual removal of the rationing system for housing in countries like Yugoslavia and Bulgaria together with the other two sensitive commodities – food and energy, allowed people's incomes to somewhat catch up with rental rates and made apartments affordable. Prices being freed, a large number of people have opted to buy apartments of their own, while the construction of new buildings has increased dramatically thus improving the terms and living conditions for tenants. Freeing of the rental market has led to substantial consumption and production gains in greater number and quality of real estate and the elimination of deadweight social loss in waiting time while queuing and in bribery. The immediate substitution effect has been that people switched away from more expensive premises to cheaper ones in towns and cities where rents are substantially lower. The liberation of the real estate market in post-communist countries has also had an income distribution effect allowing people investing in real estate to accumulate essentially more than non-owners.

Steven Cheung (1969, 1970, 1974, 1983) studied the forms of contract that shape the rental relationship between a tenant and a landlord. He analyzed different types of leasing contracts aimed at reducing transaction costs, which result from incomplete contracts. He specifically investigated the rental of land. Other scholars of rents and rental relationships include Eekhoff (1981), Schlicht (1983) and Börsch-Supan (1986).

It is considered that markets for the rental of machines are underdeveloped which is why firms typically buy their capital equipment. To compute the implicit rental rate we need to see how much it costs to buy the machine at the beginning of the period and sell it at the end of it (Varian, 1996, 318). The paper aims to study how market price of capital and rental rate are interrelated. In particular, it investigates the dynamics of the price of a machine as a result of the expected rental rate. It may be that for certain types of equipment the rental rate is relatively constant whereas for others it varies. Thus two cases are considered – one where the rental rate is expected to be constant and one where market participants anticipate it to change with time. Furthermore, the case of a variable rental rate is presented with the help of a specific rent function that increases or decreases with time. The paper is organized as follows: Section I is an introduction. Section II discusses the general relationship between rental rate and the market price of capital showing rental market equilibrium. Section III analyzes the time path of the price of capital in relation to a rental rate expected to be constant. Section IV presents the case of rental rate that increases or decreases with time.

II. Rental Rate and the Market Price of Capital

Suppose that a machine can be rented for a given time period. The rental rate gives the opportunity cost of the machine, that is, it accounts for the opportunities forgone by using the machine or self-renting it instead of renting it out to someone else. Therefore, the rental rate for a machine in its best alternative use is the cost of one machine-hour and can be denoted by v[5] (Nicholson, 1992, 330). Generally, the rental rate is seen in economic literature as the full economic price of capital used in the production process. Thus, the rental rate as the cost of a machine happens to be part of the total costs of production.

If the machine rental market is perfectly competitive, no long-run profits can be earned by renting machines. Then the rental rate v as the cost of renting the machine would be exactly equal to the cost of using it. To the owner these costs involve the sum forgone in alternative investment or the interest rate r as well as the depreciation cost of the machine to the amount of d. Thus

$$v = P(r + d) \tag{1}$$

that is, under perfect competition the rental rate will reflect both depreciation costs and opportunity costs of alternative investments where P is the price paid for the machine. Higher interest rate r and rate of depreciation d will result in higher rental rate to account for the opportunities forgone of a better investment or the quicker wearing out of the machine, respectively. The rental rate is also the marginal revenue product of the machine where a profit-maximizing firm will pay for a machine a price equal to its marginal product times the revenue generated by producing one additional unit with it.

In the private case when there is no depreciation or the machine is infinitely long-lived, the machine will resemble a perpetual bond or an indestructible capital asset where the present value of a perpetual stream of revenue from renting the machine is a constant v per year discounted at an annual interest rate of r (Chiang, 1984, 464). The present value of that revenue will be exactly equal to the market price of the machine.

$$P = \frac{v}{r} \tag{2}$$

We shall assume in our model that machines are not long-lived and deteriorate with time. For convenience we also presuppose that both the interest rate r and depreciation d are constant.

In the more general case the rental rate on machines is not constant over time. It can be assumed to be a function of time such that $v(s)$ is the rental rate of a new machine at any time s. It can also be assumed that the machine depreciates exponentially at the fixed rate d, which differs from the traditional accounting depreciation, as it is the depreciation that affects the machine's productivity.

In year s the net rental rate on a machine bought in a previous year t would be

$$v(s)e^{-d(s-t)} \tag{3}$$

where the machine has been decaying for $s - t$ number of years.

If the firm is considering buying the machine when it is new in year t, it should discount all of these net rental amounts back to that date. The present value of the net rental in year s discounted back to year t is

$$e^{-r(s-t)}v(s)e^{-d(s-t)} = e^{(r+d)t}v(s)e^{-(r+d)s} \tag{4}$$

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[5] We follow the notation used by Nicholson (1992, 330).

where r is the interest rate. Again, $s-t$ number of years has passed since the purchase of the machine until the moment the net rental is received. Therefore, the present discounted value of a machine bought in year t is the sum (integral) of all these present values. This sum should be taken from year t when the machine was bought to all years in the future. This leads to the following improper integral

$$PDV(t)=\int_t^{\infty} e^{(r+d)t} v(s) e^{-(r+d)s} ds \quad (5)$$

In equilibrium the price of the machine in year t $P(t)$ will be equal to this present value such that $PDV(t)=P(t)$ or

$$P(t)=\int_t^{\infty} e^{(r+d)t} v(s) e^{-(r+d)s} ds \quad (6)$$

Differentiating with respect to t, while using the product rule, we come at

$$\frac{dP}{dt}=(r+d)e^{(r+d)t}\int_t^{\infty} v(s)e^{-(r+d)s} ds - e^{(r+d)t}v(t)e^{-(r+d)t} = (r+d)P(t)-v(t) \quad (7)$$

Thus we obtain

$$v(t)=(r+d)P(t)-\frac{dP}{dt} \quad (8)$$

which resembles the equilibrium value of the rental rate in a perfectly competitive market shown in (1) (Nicholson, 1992, 721).

The rate of change $\frac{dP}{dt}$ in equation (8) represents the capital gains that accrue to the owner of the machine. If its price is expected to rise ($\frac{dP}{dt}>0$), the owner may accept less than $(r+d)P$ for its rental.[6] If the price of the machine is expected to fall ($\frac{dP}{dt}<0$), the owner will require more in rent. If the price of the machine is expected to remain constant over time, the derivative will tend to zero and equations (1) and (8) are identical. The conclusion is that there is a definite relationship among the price of a machine at any time, the stream of future profits that the machine promises and the current rental rate for the machine. In deciding to rent it out the owner of the machine will consider its full economic price, which involves not just its current price but also the rate of depreciation, the interest rate and the projected price changes.

The differential equation in (8) can be rewritten in the normalized form

$$\frac{dP}{dt}-(r+d)P(t)=-v(t) \quad (9)$$

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[6] Rental houses in suburbs with rapidly appreciating prices rent for less than the landlord's actual costs because the landlord also gains from price appreciation.

We can, further, solve for the market price of the machine, given the current rental rate is known where both the price and the rental rate are functions of time. Thus, we can study a reverse effect – how the rental rate affects the price of capital $P(t)$. Two cases can be examined – one where the rental rate changes dynamically and one where it is a constant function of time. The solution will help analyze the behavior of the price of capital in relation to rental rate changes.

III. The Case of a Constant Rental Rate

Assuming that the market rental rate is constant with time, we can investigate how the market price of the machine varies with time. This would be the case where the rental market has been relatively steady for a long time and firms do not expect the rent to change. It may also be true for specific types of equipment or could be applied to real estate assets. Even though the model studies the rent of machines the rents that owners of some types of real estate property receive may be relatively constant in time. Then equation (9) becomes

$$\frac{dP}{dt} - (r+d)P(t) = -v \qquad (10)$$

It is an equation with a constant coefficient and a constant term the solution of which is

$$P(t) = [P(0) - \frac{v}{r+d}]e^{(r+d)t} + \frac{v}{r+d} \qquad (11)$$

The solution shows that the market price of a machine to be rented at a constant rate will tend to diverge from its intertemporal equilibrium level $\bar{P} = \frac{v}{r+d}$. The movement of the price of capital thus shows dynamic instability. At the initial moment $t = 0$, the price approaches the intertemporal equilibrium and we obtain

$$P(t) = P(0) \qquad (12)$$

With the passage of time or as $t \to \infty$, the price $P(t)$ deviates from its initial level or equilibrium level.

Figure 2

Dynamic instability of the market price of capital with a constant rental rate

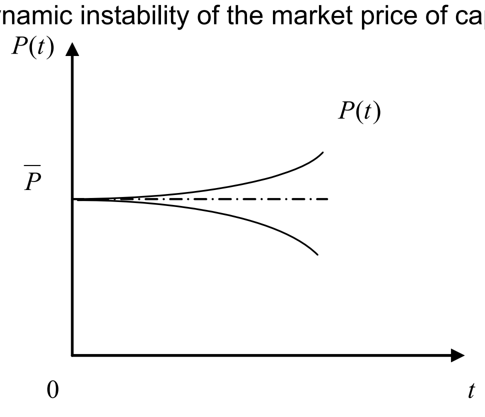
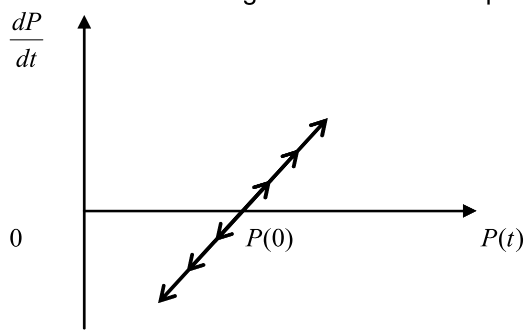


Figure 2 shows that market price can diverge from the equilibrium from above and from below. If the initial price is greater than the equilibrium level, that is $P(0) > \frac{v}{r+d}$, $P(t)$ diverges from this equilibrium from above and becomes infinitely high as $t \to \infty$. If the initial price is somewhat lower than the equilibrium level such that $P(0) < \frac{v}{r+d}$, the price of the machine diverges from below the equilibrium level. Given a constant rental rate the market participants would expect an initial market price of capital higher than the equilibrium level to grow further and an initial price lower than the equilibrium level to continue to fall.

Figure 3

Divergence of the market price of capital



The time path of $P(t)$ can also be represented with the help of a phase diagram. Figure 3 shows $P(0)$ as the intertemporal equilibrium where the rate of change $\frac{dP}{dt}$ is zero and at any other moment t price is dynamically unstable. At the point of dynamic stability equation (10) translates into

$$P(0) = \overline{P} = \frac{v}{r+d} \tag{13}$$

or the equilibrium price is exactly equal to the ratio of the rental rate and the sum of the opportunity costs and the depreciation costs of the machine. If firms expect the market rental rate to be relatively steady, they will pay for a machine exactly this ratio at the equilibrium. With time the price will be likely to deviate from this initial equilibrium and show dynamic instability given a constant rental rate.

It should also be noted that the slope of the phase line is given by the sum $r+d$, which is positive. We presumed that the interest rate and depreciation take constant values. However, as those values are chosen to be higher, the slope would be steeper, showing that the price $P(t)$ will be growing or declining exponentially faster, thus, diverging faster from the initial level. For smaller values of r and d the divergence will be slower. If the interest rate as the opportunity cost of a machine and the rate of economic depreciation are lower, market participants will not foresee strong divergence of price from its equilibrium level.

IV. The Case of a Variable Market Rental Rate

It is more realistic to assume that the market rental rate changes with time such that equation (9) shows the time path of the price of capital as the rate at which it can be rented is expected to vary. The equation then becomes one with a constant coefficient and a variable term. The general solution to a differential equation with a variable coefficient and a variable term of the type

$$\frac{dy}{dt} + u(t)y = w(t) \tag{14}$$

is

$$y(t) = e^{-\int udt}(A + \int we^{\int udt}dt) \tag{15}$$

We use the formula for our specific case of a constant coefficient such that

$$P(t) = e^{-\int -(r+d)dt}\left[A + \int - v(t)e^{\int -(r+d)dt}dt\right] = e^{(r+d)t}\left[A - \int v(t)e^{-(r+d)t}dt\right] \tag{16}$$

where A is an arbitrary constant that can be definitized if we have an appropriate initial condition. This means that A can be found at $t = 0$. To find A we need a specific form of the rental rate function of time $v(t)$ in expression (16).

4.1. The Rental Rate as a Growing Function of Time

If the rental rate is expected to grow with time such that $\frac{dv}{dt} > 0$, we can find how that affects the price of capital $P(t)$. For simplicity let us assume that $v(t) = t$, that is, the rental rate is exactly the same as the time period and, therefore, grows with time.[7]

Then we can substitute for $v(t)$ in equation (16), where using integration by parts we solve for $\int te^{-(r+d)t}dt = -\frac{te^{-(r+d)t}}{r+d} - \frac{e^{-(r+d)t}}{(r+d)^2}$. This yields

$$P(t) = e^{(r+d)t}\left[A - \int te^{-(r+d)t}dt\right] \tag{17}$$

Further integration gives the general solution for the price of capital under the assumed specific form of the rental rate function

$$P(t) = e^{(r+d)t}\left[A + \frac{te^{-(r+d)t}}{r+d} + \frac{e^{-(r+d)t}}{(r+d)^2}\right] = Ae^{(r+d)t} + \frac{t}{r+d} + \frac{1}{(r+d)^2} \quad \text{for } v(t) = t \tag{18}$$

[7] This is an oversimplification as it implies that a machine is not rented out in the initial moment at $t = 0$ as $v(t) = 0$. But the function is convenient to use for the purposes of our analysis.

To find the definite solution for $P(t)$ we need to find the value of A at the initial moment $t=0$. Hence,

$$P(0) = A + \frac{1}{(r+d)^2} \qquad (19)$$

$$A = P(0) - \frac{1}{(r+d)^2} \qquad (20)$$

Substituting for A in equation (18) we obtain the specific solution of price of capital.

$$P(t) = \left[P(0) - \frac{1}{(r+d)^2} \right] e^{(r+d)t} + \frac{t}{r+d} + \frac{1}{(r+d)^2} \text{ for } v(t) = t \qquad (21),$$

where $r, d, t, P(0)$ are all positive.

We use this result to analyze how the price of capital would change with time. As time passes ($t \to \infty$) price will be growing or falling exponentially depending on whether the parenthesized expression is positive or negative. Price $P(t)$ will be growing exponentially if $P(0) > \frac{1}{(r+d)^2}$ and falling exponentially when $P(0) < \frac{1}{(r+d)^2}$. As time goes by and as the rental rate for a machine is expected to grow, the price of the machine will increase or decrease exponentially depending on the level of the initial price.

4.2. The Rental Rate as a Decreasing Function of Time

What if our projections are that the rental rate will be falling with time such that $\frac{dv}{dt} < 0$? How would that affect the price of capital $P(t)$? This time we would assume that $v(t) = -t$ and the rental rate takes the negative value of the given time period and, therefore, decreases with time.

We, again, substitute for $v(t)$ in expression (16)

$$P(t) = e^{(r+d)t}\left[A + \int te^{-(r+d)t}dt\right] = e^{(r+d)t}\left[A - \frac{te^{-(r+d)t}}{r+d} - \frac{e^{-(r+d)t}}{(r+d)^2}\right] = Ae^{(r+d)t} - \frac{t}{r+d} - \frac{1}{(r+d)^2}$$

for $v(t) = -t$ (22)

Expression (22) of the rent decreasing with time very much resembles expression (21) where it is increasing gradually.

Similarly to the first case, we can find the arbitrary constant A for the definite solution of (22).

At moment $t=0$ equation (22) becomes

$$P(0) = A - \frac{1}{(r+d)^2} \tag{23}$$

or

$$A = P(0) + \frac{1}{(r+d)^2} \tag{24}$$

Thus, the definite solution for the price of a machine as the rental rate is expected to diminish, becomes

$$P(t) = \left[P(0) + \frac{1}{(r+d)^2} \right] e^{(r+d)t} - \frac{t}{r+d} - \frac{1}{(r+d)^2} \quad \text{for } v(t) = -t \qquad (25),$$

where $r, d, t, P(0)$ are all positive.

The analysis of this final result shows that as $t \to \infty$, so does $P(t)$. It grows exponentially, while the rental rate decreases with time.[8] For some very large values of t, the price of capital is expected to approach infinity. This occurs for any value of the initial price $P(0)$.

Conclusion

While the traditional approach studies how the rate at which a machine can be rented depends on the market price of that machine, it is interesting to trace the time path of the price of capital in relation to a given expected rental rate. An easy case is when the rental rate is relatively stable and firms do not expect it to change with time. Under this scenario the intertemporal equilibrium market price of the machine is the initial price. With time the price diverges from this intertemporal equilibrium.

When market participants expect the rental rate to increase with time, the price of the machine will increase or decrease exponentially depending on the initial price level. Given that rental rate is expected to fall, the market price of capital will grow exponentially.

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[8] The first exponential term in the equation grows faster than the second one falls by the amount of t.

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