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ON THE UTILITY OF MONEY

The paper studies the concept of utility of money. The latter is defined as the ability to generate additional income. Utility of money is maximized by equalizing dynamic marginal utility of money and marginal cost. The subsequent differential equation links up income velocity of money acceleration and equilibrium convergence. It is proved that individuals' utility of money maximization objectives may be aggregated at macro level and can be used for monetary policy optimization. Lagrangian multipliers technique is applied to obtain a relationship between income velocity of money and some supplementary constraints. The paper also makes distinction between short term and long-term utility of money. Conclusions about different types of monetary policies are derived. The Appendix establishes connections with production and investment on the basis of optimal control technique.
JEL: E00; E31; E51

1. Brief Description

The utility of money is one of the most complicated economic problems. On the one hand, money plays crucial role in economic exchange, but, on the other, it does not have intrinsic utility. Consequently, the utility of money must be derived from its systemic functions.

In particular any conjecture about the utility of money have to be built on the classical utility theory, the theory of utility of monetary income plus additional assumptions, related to the economic functions of money. The most important of these functions are the medium of exchange, the numéraire and the store of value.

The paper starts with brief exposition of utility problem and Roy's identity. Next it proceeds with the introduction of exchange and an augmented Clower constraint, based on stylized monetary exchange economy.

The key element allowing for construction of utility of money function is the idea of Weitzman to use money metric and net material product respectively as measure of present and future consumption.

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Furthermore, since the role of money is to facilitate exchange, the utility of money is derived from its ability to generate additional income compared to the situation with double coincidence of wants, i.e. the state of exchange without money. Utility of money is reduced to the solution of dynamic second order differential equation with inflation and income velocity of money as parameters.

Next, the long and short-run aspects of the utility of money are studied as well as additional constraints and optimization. Velocity of money is related to circulation time and Lyapunov exponents.

Finally the connections to production and investment are established on the basis of optimal control technique.

2. Roy's Identity, Clower Constraint and Net National Product Concept

To start with utility theory, let assume an economy populated with individuals maximizing additive utility. The problem every consumer faces can be stated as follows: to find point x^* in R^{n+} with the property that x^* satisfies the budget constraint so that $p_1 x_1^* + p_2 x_2^* + ... p_n x_n^* \leq m$ and $x^* \geq x$.

In spite of directly maximizing utility as function of prices and subject to budget constraint, we can view the quantities as function of prices and utility as function of quantities or $U(p,m) \equiv u[x(p,m)]$. The second function is known as indirect utility function.

Using the so called Envelope Theorem and Lagrangian multipliers technique one can deduce the well known Roy's identity: $\partial U / x_i \partial p_i = \partial U / x_i \partial m = -\lambda$.[2] In this case λ is interpreted as the marginal utility of income.

The Roy's identity tells us that partial derivatives of individual utility functions with respect to prices per unit of any good are equal to each other and match the marginal utility of monetary income.

The Roy's identity is a useful initiation of our investigation, but it clearly does not resolve the problem of the utility of money while conveniently emphasizing the significance of the utility of monetary income.

Obviously the monetary income itself should be in the focus of our study since it's the final product of the circulation of money. However we need to introduce explicitly exchange, money and their relationship.

[2] Takayama, A. Mathematical Economics. Second Edition, 1990, p. 139.

The exchange additionally complicates the problem, since the economic agents are supposed not only to maximize utility of consumption under budget constraint, but to exchange their endowments for their final demand bundle of goods. Here money plays crucial role.

The monetary economy eliminates the double coincidence of wants[3] constraint (Jevons, 1875)[4], thus broadening the range of the possible exchanges. On the other hand, since the counterparties' bilateral exchange objectives do not match each other any more, all information is not revealed and additional interactions are necessary to attain general equilibrium. The medium of exchange, transmitting signals from one agent to another, becomes an indispensable instrument of removing market misalignments. Intuitively, the speed of such transmission is crucial in terms of equilibrium convergence.

The elimination of Jevons' double coincidence of wants (DCW) constraint is possible if money can integrate and coordinate the decentralized exchanges, i.e. if the circulation and its velocity respectively, posses some systemic properties.

To resolve the problem we assume monetary economy with Clower rule – "money buys goods and goods buy money; but goods do not buy goods".[5] We impose however additional requirement. Money buys financial instruments (including deposits) and financial instruments buy money, but financial instruments never buy goods or other financial instruments.[6] In other words the (base) money is the only financial instrument fulfilling all the three basic functions – medium of exchange, numéraire and the store of value.

Such augmented Clower constraint shrinks all financial instruments to different types of money substitutes and allows for construction of stylized financial system reduced to banking system. In addition, these rules imply that any change in the real economy has monetary dimensions. To describe such an economy we need further elaborations.

The elimination of the DCW rule means simultaneous removal of interpersonal and temporal constraints on exchange. That is, any non-DCW trade rule implies saving- one of the participants, namely the seller, does not obtain immediately the goods and services he or she needs, but at least for the interim he or she "saves" the income. The duration of saving may be very short.

The other side in exchange, the buyer, either uses its prior savings, or obtains financing from a third part – the banking (financial) system. Prior savings may exist only if institutions issuing money had already being in place. So at the origin of any

[3] Without money every pair of economic agents can exchange goods if and only if the first needs what the second offers and vice versa.

[4] Jevons, W. S. Money and the Mechanism of Exchange. London: Macmillan, 1875.

[5] Clower, R. A Reconsideration of the Micro Foundations of the Monetary Theory. – Western Economic Journal, 1967, 6(4), p. 1-8.

[6] This means, for example, that financial instruments such as bills of exchange are either not used or are immediately discounted with the banking system.

monetary exchange we need implicitly three types of agents – seller (also saver-lender), buyer (also borrower-spender) and financial intermediary (bank).

This mechanism entails also combination of functions of money as store of value, medium of exchange and numéraire and allows for the elimination of the DCW rule via the mechanism of generating saving and borrowing. This allows us to put up a simple economy with banking system supplied money.

This approach to money and credit differs from both exchange credit and asset capital approaches.[7]

We unify money and credit since both diverge only in details (duration) while sharing the same origin (deposits and loans are based on the circulation of currency via the multiplier). The other financial assets are also derived from money as far as they must be swapped for cash to fulfill their functions.

Introducing money and financial instruments means that budget constraint of individuals, maximizing utility becomes dynamic. It depends on quantity of goods and financial instruments individuals can sell. Selling financial instrument (selling bond or obtaining loan) means that economic agent augment present spending, while buying bond or opening deposit reflects postponement of spending and consumption. Technically this means that utility function includes future consumption and prices contain discounting factor (interest rate).[8] The Roy's identity obtains the following form $\partial U_t / x_t^i \partial p_t^i = \partial U_t / x_t^i \partial m_t^i = -\lambda_t$.

After introducing our simplified financial system, we can proceed with the monetary income.

Since the seminal paper of Weitzman (1976)[9] we know that it is possible to establish connection between utility (formulated as utility of consuming bundle of goods) and monetary Net National Product (NNP or simply income). Under certain general

[7] Gillman, M., P. L. Siklos, J. L. Silver. Money Velocity with Costly Credit. Draft prepared for the 1997 European Economic Association Meetings, 1997.

[8] The prices of the financial instruments are numbers, equalizing the face value of the instrument and its market price. Suppose the face value of the instrument (deposit) is x and its period is one year. Its' net present value (not including interest r) is $x_p = \frac{x}{(1+r)}$. The price p is $(1+r)$. The product $px_p = x$. So we can define the broad money as product of vectors of prices and NPV-es. When the interest is effectively calculated it's added to the deposit amount.

[9] Weitzman, M. L. On the Welfare Significance of National Product in a Dynamic Economy. – Quarterly Journal of Economics, 1976, Vol 90, p. 156-62.

conditions[10], the utility unit is immaterial and applying money metric does not distort welfare analysis.

The utility of monetary income is simply the utility of present and future consumption that can be obtained by exchanging money for real goods and services.

In particular, it can be proved that, at least locally, an increase of monetary NNP implies an increase in utility.[11] Building on this, Li and Löfgren had proved that in the case of local or exogenous growth, the monetary NMP measure and the other more complex measures give the same results.[12]

It is assumed, as a rule, that possession of liquidity reduces the transaction costs of trading consumption goods only, without impact on capital goods (see among others Bosi[13] and Dotsey and Sarte[14]). Provided that long-term funds and money markets are interdependent, it is not legitimate to confine the utility of money to consumption only. Moreover, it seems that the non DCW rule is more important just in case of capital goods. Accordingly we deduce that money facilitates all exchanges.

Following this logic, the utility of money is defined in this paper as the additional monetary income that can be derived by eliminating DCW rule. Given the augmented Clower constraint imposed on our economy, just any increase of monetary income indicates positive utility of money. Following this logic, and taking into account that monetary income reflects present and future consumption, we can conclude that by maximizing utility of money we maximize present and discounted future consumption.

We must stress that our understanding of utility of money is based on the functions of money as numéraire, medium of exchange and store of value. Other functions or motives such as precautionary and speculative are not explicitly taken into account. While this omission makes some conclusions probably less realistic, it does not mean really loss of generality in conceptual terms since both precautionary and speculative motive presuppose the use of money as medium of exchange in the future. In terms of utility this is postponed, expected or conditional transaction utility.

[10] These conditions are related to prices, investment and production and are discussed in the Appendix.
[11] Asheim, G. B., M. L. Weitzman. Does NNP Growth Indicate Welfare Improvement? Memorandum 2/2001, Department of Economics/University of Oslo, 2001.
[12] Li, C. Z., K. G. Löfgren. On the Choice of Metrics in Dynamic Welfare Analysis: Utility versus Money Measures. February 2002.
[13] Bosi, S. Money, Growth and Indeterminacy. EPEE, University of Evry, 12 March 2001.
[14] Dotsey, M., P. D. Sarte. Inflation Uncertainty in a Cash-in-Advance Economy. – Journal of Monetary Economics. 2000, Vol 45, p. 631-55.

We begin with the general specification of the money in the utility function[15], our objective being not simply to “add” the utility of money to the utility of consumption, but to express the total utility in dynamic monetary terms:

(1) $U_t = u(c_t, z_t)$

Where z_t represents the flow of services yielded by money and c_t is the consumption in time t. Since the monetary income is a function of present and future consumption and since the services of money clearly depend on the quantity of money and income, we can rewrite the equation (1) as follows:

(2) $U_t = u[c_t(y)_t, z_t(M_t)] = u^*(y_t, M_t)$

Where y_t and M_t are the income and the quantity of money.

More precisely, we can define the ***local utility of money*** as the monetary income accretion, which can be obtained by augmented selling of goods and financial services. The local utility of money equals the first derivative of the marginal utility of monetary income of the Roy’s identity with respect to time, or $u^* = \lambda_t'$.[16]

In addition, we can define ***the marginal local utility of money*** as the supplementary monetary income that can be generated by ***accelerating*** the exchange of goods for the money in circulation. The marginal local utility of money equals the second derivative of the utility of monetary income or $u^*' = \lambda_t''$.

We assume also the existence of some equilibrium or maximum level of (real) monetary income and consequently existence of local maximum. Deviations from equilibrium, caused by external shocks, trigger dynamic convergence process. The economic agents are assumed to be rational and to attain equilibrium by equalizing marginal utility and marginal cost.

The next step is to argue that the economic agents may simply maximize the utility of money as medium of exchange. In other words, the objective to maximize monetary income should be equivalent to the objective of maximizing the utility of money. In turn, maximizing monetary income via (modified) Roy’s identity is equivalent to maximizing present and future consumption.

Since the ***local utility of money*** is defined as the ***additional monetary income*** that can be generated, then at any point of time it coincides with the speed of convergence to equilibrium.

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[15] Walsh, C. E. Monetary Theory and Policy. Cambridge: Massachusetts Institute of Technology Press: 2003, 612 p.

[16] In this case we assume consumption as dynamic process.

So we can write:

(3) $U_t = \bar{y}_t{}' P_t$

We define $\bar{y}_t$ as $\bar{y}_t = y^* - y_t$, where y^* is the real equilibrium NNP, y_t is the current real NNP and P_t is the price level at moment *t*. The NNP can be both the net income of any individual agent or the NNP at national level. In other words (3) gives us the utility of money, given the current price level. Since obviously $\bar{y}_t^{'} = y_t^{'}$ we can rewrite (3) as:

(4) $U_t = y_t^{'} P_t$

So, the local utility of money equals the speed of convergence to equilibrium at a given price level. By utility of money we denote local utility unless different meaning is explicitly stated.

If the current NNP is not expressed in equilibrium prices, the deviation from the equilibrium NNP can be both positive and negative (obviously, if both current and equilibrium income are expressed in equilibrium prices, the difference is positive by definition). We'll see later that the assumption about equilibrium prices is not substantial.

Nevertheless it is obvious from (4) that the local utility of money can be both positive and negative. This creates problem, since the total utility must be always non-negative by definition. Nevertheless locally money, in non-equilibrium prices, could have negative utility if the current income exceeds the equilibrium one.

Following our logic, the marginal utility of money in current prices is given by the following formula:

(5) $U' = \bar{y}'' P_t$

Were U' is the ***marginal local utility of money*** or the additional utility that can be derived by ***accelerating*** the exchange of goods; $\bar{y}''$ is the second derivative of the real income deviation from equilibrium with respect to time.

We must distinguish between ***long run utility*** (always non-negative) and ***local utility of money*** (maybe negative)**.**

To obtain the long run utility (utility in the large) we must reformulate the concept of disequilibrium. In the short run the disequilibrium is some deviation from the optimum. In the long run we can view the disequilibrium as increasing deviation from some initial point, reflecting the process of economic growth, so we have:

(6) $U_L = U_L(\bar{y}) = U_L(y_t - y_0^*)$

If the economic growth takes the form of $y_t = y_0^* e^{rt}$, we can define the utility of money in the long run as:

$$(7)\ U_L = \sum_{n=1}^{\infty} \iint_{t_n} \beta_t P_t y_{t_n}^{'} \partial y_{t_n} \partial t + \int_0^{\infty} [\beta_t \sqrt{y_0^*(e^{rt} - 1)}]\partial t$$

Where β is discounting factor of the type $\beta_t = e^{-st}$. The first term of (7) reflects the utility of money, resulting from external shocks ($n = 1,2,3...\infty$) disturbing the economy from the respective local equilibrium states $y_{t_n}^*$ and the second term replicates long term utility of money. So we can write down:

$$(8)\ U_L = \sum_{n=1}^{\infty} \iint_{t_n} \beta_t P_t y_{t_n}^{'} \partial y_{t_n} \partial t + \int_0^{\infty} [\sqrt{y_0^*(e^{t(r-2s)} - e^{-2ts})}]\partial t$$

Depending on whether the growth factor *r* is bigger, equal or lower then $2s$, the second part of (8) is infinitely high or zero. The first part is always positive, so (8) meets the non-negativity constraint of the long-run utility of money. In other words, (8) confirms that our concept of utility of money is consistent in the long run, so we can focus on short-run equilibrium convergence.

3. Equilibrium Convergence

To solve the problem of agent's based equilibrium convergence, we acknowledge that position of every rational economic agent is dual.

On the one hand, she or he tries to exchange its products for money to obtain desired pecuniary income. This exchange comprises also selling of financial instruments (borrowing) as far as such operation is equivalent to selling products on futures markets. The financial intermediaries (banks) are supposed to sell financial services and be part of the real sector. The only outputs of the banking sector are the intermediary services.[17]

On the other hand, the economic agents spend money to obtain the necessary products. These products include not only the present, but also future consumption. In other words spending includes buying financial instruments (lending).

[17] Wang, C. J. Loanable Funds, Risk, and Bank Service Output. Research Department, Federal Reserve Bank of Boston, July 2003, p. 2.

Proposition 1: Maximizing the local utility of money (maximizing the speed of convergence of NNP to equilibrium) is necessary and sufficient condition for maximizing the cumulative utility of attaining equilibrium.

Proof:

1. Necessity

Necessity means that increasing monetary income at the equilibrium path we increase in the same time the utility of money.

The utility of money is defined as the speed of decline of the deviation from equilibrium, multiplied by the price level.

We express the local utility of money at current price level as $U = \bar{y}' P_t = [\partial(y_t - y^*)/\partial t] \cdot P_t = [\partial y_t / \partial t] \cdot P_t = y_t' P_t$. The cumulative utility of the overall convergence period in fixed prices ($P_t = 1$) is: $U = \int_{y_{t0}}^{y_{t^*}} \beta_t y_t' \partial y_t$,

where β_t is the discounting factor, decreasing with the duration of the convergence period.

So if the growth rate of the real income increases, the local utility of money increases too; the higher the initial divergence from equilibrium, the bigger the cumulative utility of money. The cumulative utility increases with the acceleration of convergence as well, since the discounting factor declines. At equilibrium the local utility of money and equilibrium convergence speed equal zero.

2. Sufficiency

Sufficiency indicates that by increasing or decreasing the utility of money we affect in the same direction the speed of convergence of NNP and that attaining maximum cumulative utility of money we attain local monetary income maximum.

From (3) and (4) it follows that in equilibrium the local utility of money is zero. Under the obvious condition $P_t > 0$ it results also that $\bar{y}' = 0$.

Diverging from equilibrium we have positive utility of money so we can deduce that $U > 0$ and $y_t' > 0$. Since local utility of money decreases and income increase we have $U' < 0$ and $y_t'' < 0$. This implies that income attains local maximum. In the same time the cumulative utility of money also reaches its upper limit.

This proves the sufficiency.

The Proposition 1 means that maximizing utility of money and maximizing monetary income are dynamically equivalent strategies.

The equilibrium conditions imply that the marginal utility, resulting from the generation of additional pecuniary income, equals the marginal loss of (money) utility from spending money.

Since the utility of money is based on its function of store of value via increasing monetary income and cash balances, the disutility must be defined as a loss of this property via spending money. This approach is just the opposite of the traditional interpretation, assuming utility for the buyer and disutility for the seller (see Trejos and Wright, 1995). Nevertheless it is the only reasonable position since we maximize the utility of money.

On micro level the utility of money of agent i is measured as $\overline{y}_i^{'} P_t$ and the marginal utility as $\overline{y}_i^{''} P_t$. We assume that increasing supply of goods in order to generate additional income implies increasing costs in terms of increased purchases of additional inputs or financial instruments (future income).

Increased purchases mean accelerated exchange of money for inputs and consequently higher income velocity of money. Therefore, the change in the marginal nominal disutility of money must take into account the variation in the income velocity of money and the opportunity costs in terms real income foregone.

The first marginal disutility term may be defined in the following way. Observe that we can express the inter-agents cross velocities of money as functions of incomes deviations from equilibrium. The differential of total agents' velocity of money can subsequently be expressed in the following way: $dv_i = \frac{\partial v_{i1}}{\partial \overline{y}_{i1}} \overline{y}_{i1} + \ldots \frac{\partial v_{ii}}{\partial \overline{y}_{ii}} \overline{y}_{ii}$.[18] For a single point of time this expression can be transformed into $dv_i = v_{ij}^{'} \cdot \overline{y}_{ij}^{'}$. Taking into account the price level we can express the disutility term as $U_{MD_i 1}^{'} = [(v_{ij}^{'} \cdot \overline{y}_{ij}^{'}) \cdot P_t]$, where the expression $(v_{ij}^{'} \cdot \overline{y}_{ij}^{'}) \cdot P_t$ is a product of the velocity acceleration vector $v_{ij}^{'}$ [19] and the income acceleration vector multiplied by the price level. The term is interpreted as the marginal disutility of spending money in nominal terms.

[18] Here and later we omit the symbol "0" from v_{ij}^{0} for simplicity. We assume always base money unless different interpretation follows from the text.

[19] The velocity acceleration vector of the agent i is defined in this case as the acceleration of money velocity between i and the other agents. The velocity is understood as the net value added divided by the quantity of money the respective agent holds.

We must also add to marginal costs additional term, capturing income depreciation. This second marginal disutility term equals the product of price level shift and income divergence from equilibrium $P'\overline{y}_i$. Or $U'_{MD_i 2} = P'\overline{y}_i$. In particular, the expression $P'_t\overline{y}_i$ equals the cost (benefit) in terms of additional monetary income, necessary to purchase $\overline{y}_i$ at higher (lower) price level.

The assumed rationality of economic agents implies equality of marginal pecuniary revenue and marginal cost:

(9) $$U'_{Mi} = U'_{MD_i 1} + U'_{MD_i 2} = \overline{y}''_i P_t = (v'_{ij} \cdot \overline{y}'_{ij}) \cdot P_t + P'_t \overline{y}_i$$

Dividing equation (9) by P_t we obtain a second order homogenous differential equation in $\overline{y}_i$.

(10) $$\overline{y}''_i - v'_{ij} \cdot \overline{y}_i' - \frac{P'_t}{P_t}\overline{y}_i = \overline{y}'' - v'_{ij} \cdot \overline{y}'_i - p\overline{y}_i = 0$$

Where p is the inflation rate ($p = \frac{P'_t}{P_t}$). The homogeneity of the differential equation (10) reflects the rationality of the economic agents' behavior (parity between marginal utility and marginal cost).

Equation (10) reflects the individual utility function maximization.[20] At macro level (with additive utilities), we have:

(11) $$\sum_{i=1}^{n}(\overline{y}''_i - v'_{ij} \cdot \overline{y}'_i - p\overline{y}) = \overline{y}'' - V' \cdot \overline{y}' - p\overline{y} = \overline{y}''_m - v'(\overline{y}'_1 + \overline{y}'_2 + ..\overline{y}'_n) - p\overline{y}_m = \overline{y}''_m - v'\overline{y}'_m - p\overline{y}_m$$

Where $\overline{y}_m$ is the aggregated deviation[21] from equilibrium income (NNP) and v' is the (first derivative of) the income velocity of money at macro level.[22] Further in the

[20] In the second part we introduced three types of economic agents – seller (also saver-lender), buyer (also borrower-spender) and financial intermediary (bank). However, based on the utility of money we can replace all three types by a single monetary description, the latter being sufficiently abstract.

[21] When there is macroeconomic disequilibrium equation (11) is correct. The real difficulty is to explain monetary income generation under macro equilibrium. In this case the individual $\overline{y}'_i$ and $\overline{y}''_i$ as well as marginal velocities must be taken in absolute terms. This reflects disequilibrium transmission function of money. In such a case velocities and income changes have sporadic local nature similar to quantum processes in physics.

text the subscript m is omitted and $\overline{y}$ is assumed to be the deviation of income from equilibrium at macro level.

Here the income velocity of money and its first derivative have different structural meaning compared to traditional Fisher's equation of exchange. The income velocity is not simply the ratio between income and the quantity of money, but requires the fulfillment of condition $\det(V'-v'I)=0$. This means that circulation monetary cycles involving all agents (or, alternatively, all markets) must exist (the matrix determinant includes combination of elements of all rows and columns).

The equation (11) allows for the optimization of system at macro level if the economy is in macro disequilibrium. Under macro equilibrium (aggregate demand equals aggregate supply) money plays the role of disequilibrium transmission at micro level and the individual agents' deviations and velocities must be treated differently.

If velocity and inflation adjust instantly in the sense that they attain immediately stable rates of change until equilibrium is restored, we can view (11) as fixed coefficients second order homogenous differential equation of the standard form $y''+py'+qy$. Note that the equation (11) is of the same type as the equation describing the oscillation of load suspended by a spring along a vertical straight line in classical mechanics.[23]

As it is well known, in order to solve the equation (20), we need to write down the respective characteristics equation which in this case reduces to:

(12) $r^2 + v'r + p = 0$

Where r is the root of the equation (12).

The number of roots depends on the sign of the expression $(v')^2 - 4p$. If it's zero, we have one real root. If it is negative we have two complex roots and if positive- two real roots.

In the case when the discriminant is zero, the solution is especially simple:

(13) $\overline{y} = Ce^{rt} = Ce^{-\frac{v'}{2}t} = \overline{y}_0 e^{-\frac{v'}{2}t}$

[22] We assume that the matrix equation $V'\overline{y}'$ has a Frobenius root or $V'\overline{y}' = v'\overline{y}$ where v' is the Frobenius root. The matrix V' is considered non-negative, indecomposable and primitive. The Frobenius root is interpreted as the income velocity of money at macro level.

[23] Shipachev, V. S. Higher Mathematics (English). Moscow: Mir Publishers, 1988.

It means that whatever the deviation from equilibrium income, only an increase of velocity of money can guarantee equilibrium convergence. Consequently all types of disequilibrium imply positive growth of velocity.

From the point of view of the social optimum both overheating and underemployment are inferior solutions and presume elimination of inefficiencies. Note also that (13) does not require constant optimum real quantity of money, but implies proportionality between velocity acceleration and inflation, as defined by the discriminant, namely $v' = 2\sqrt{p}$. It means also decline of real quantity of money if nominal money supply is fixed by the authorities.

In the situation when the discriminant is negative, the solution takes a more complicated form.

$$(14)\ \ \bar{y} = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t)\,;\ \alpha = -\frac{v'}{2}; \beta = \frac{1}{2}\sqrt{(v')^2 - 4p}$$

In the case the equation (15) the increase of the income velocity of money is still a necessary equilibrium convergence condition, but the process obtains cyclical features. The latter are negatively affected by the inflation rate in the sense that the higher the inflation the shorter the cycles.

Note that (14) also does not imply constant optimal real quantity of money. It does not require even strict proportionality between velocity acceleration and inflation. In fact (14) is compatible with declining real quantity of money, if the nominal money supply is fixed. So (14) describes situation when the initial real money supply exceeds the equilibrium level.

The equation (14) can be interpreted in favor of Lucas' hypothesis of impossibility of distinguishing between inflation and relative prices changes under certain parameters of economic system dynamics- if the economic agents observe cyclical oscillations of prices and quantities around the equilibrium values and since these oscillations nay not be synchronized, it maybe impossible to separate inflation and relative prices shifts.

An additional attribute is the doubling of the initial conditions $(C_1 \quad and \quad C_2)$. The latter can be interpreted as deviation from the equilibrium in terms of quantities (Marshallian convergence) and prices (Walrasian convergence). These deviations can alternate in sign in the course of convergence. This maybe viewed as additional argument in favor of Lucas hypothesis.

The discriminant of our differential equation can be positive only if inflation is very low or negative (zero velocity acceleration excluded). This gives two distinct real roots. In this case the solution takes the following form:

(15) $\bar{y} = C_1 e^{r_1 t} + C_2 e^{r_2 t} : r_1 = \dfrac{-v' - \sqrt{(v')^2 - 4p}}{2} : r_2 = \dfrac{-v' + \sqrt{(v')^2 - 4p}}{2}$;

In principle low inflation rate implies fast equilibrium convergence process in terms of prices (presumably r_1, since the first term always converges faster then the second as far as $v' > 0$) and slow in terms of quantities (r_2 term converges in principle at slower pace). Note also that deflation rules out convergence since r_2 becomes positive.

If the nominal money supply is fixed, the equation (15) implies (under deflation) increasing real quantity of money. Therefore, deflation, increasing real money supply, accelerating velocity and equilibrium convergence[24] are not compatible simultaneously. Deflation accelerates convergence in terms of prices but generates slow divergence in terms of quantities, thus deflation could have positive impact only in short run.

To illustrate further the role of income velocity of money, let's slightly modify Fisher's equation of exchange, namely as $\widetilde{M}_t^0 v_t = \widetilde{y}_t$, where $\widetilde{M}_t^0 = M_t^0 / P_t$ and $\widetilde{y}_t = y_t / P_t$ are the real base money supply and income in constant prices. Initially the money supply, velocity and real income are below the equilibrium levels. Differentiating with respect to time we obtain $\widetilde{M}_t^{0'} v_t + v_t' \widetilde{M}_t^0 = \widetilde{y}_t'$.

Assuming Walras law (the negative excess demand on money market equals the positive aggregate excess demand on the other markets) and taking into account that eliminating disequilibrium on both markets adds to the real NNP under equilibrium convergence (or $\dfrac{\partial \widetilde{y}_t}{\partial t} = 2 \dfrac{\partial \widetilde{M}_t^{0'}}{\partial t}$), we obtain $v_t' = 2 - \dfrac{\widetilde{M}_t^{0'}}{\widetilde{M}_t^0} v_t$.

Since the velocity depends on the period for which it is calculated, for sufficiently short horizons and sufficiently small v_t, the increase of real money supply from below the equilibrium will always engender velocity acceleration, reflecting the synchronized mode of market system convergence to equilibrium.

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[24] If we assume gross substitutability among markets, then deflation implies increasing aggregate demand even under fixed nominal money supply. Since supply is always physically constrained, prolonged deflation involves increasing excess demand on all markets. This however violates Walras low and requires return to inflation and declining real demand for money.

4. Additional Constraints and Optimization

As we can generalize, the equilibrium convergence depends crucially on v' - the higher the velocity acceleration, the faster the convergence (excluding prolonged deflation). Cyclical and discriminant terms only add vanishing oscillations or parallel movements, so that for simplicity sake we can ignore them.

Finally, assuming optimization, the problem can be stated as:

Maximize v'

Subject to:

1. Cost-benefit condition, that is the marginal cost in terms of first derivative of income deviation with respect to velocity acceleration, should equal the marginal benefit, measured as first derivative with respect to time, or:

$$\partial\overline{y} / \partial v' = \partial\overline{y} / \partial t$$

$$-\frac{t}{2}e^{-\frac{v'}{2}t} = -\frac{v'}{2}e^{-\frac{v'}{2}t}$$

$$v' = t$$

This condition is needed because we assumed fixed coefficient equation but there is no explicit guarantee that v' is at optimum level. The cost-benefit term just indicates that the system converges for a fixed period of time ($v' = t$) from $\overline{y}_0$ to $\overline{y}_0 e^{-\frac{(v')^2}{2}}$ and that the convergence may not be complete.

2. Discriminant conditions ($v'^2 - 4p \geq 0$ or $4p - v'^2 \geq 0$);

3. Fisher's identity in terms of growth rates: $\dot{m} + \dot{v} = p + \dot{y}$, or for $\dot{m} \geq 0$, we can write $p + \dot{y} - \dot{v} \geq 0$.

First let solve the problem for non-negative discriminant (relatively low inflation).

Then the Lagrangian takes the following form:

$$\Phi = v' + \lambda_1(v' - t) + \lambda_2(v'^2 - 4p) + \lambda_3(p + \dot{y} - \dot{v})$$

Since under equilibrium convergence process we have $-\partial\overline{y}/\partial t = \partial y/\partial t$ and after an appropriate change of the initial conditions (namely $y_{t_0} = 0$), we obtain $\dot{y} = v'/2$. Note also that as far as we presume fixed coefficients differential equation $\partial p/\partial v' = \partial v'/\partial p = 0$, by definition.

So we get the following system of equations:

$$\partial\Phi/\partial v' = 1 + \lambda_1 + 2\lambda_2 v' + \lambda_3(1/2 - 1/v) = 0$$

$$\partial\Phi/\partial p = -4\lambda_2 + \lambda_3 = 0$$

$$\partial\Phi/\partial t = -\lambda_1 = 0$$

$$\lambda_1(v' - t) = 0$$

$$\lambda_2(v'^2 - 4p) = 0$$

$$\lambda_3(p + v'/2 - v'/v) = 0$$

Solving for v', p, λ_1, λ_2 , λ_3 *and* t we find:

$$v' = \frac{2(2-v)}{v}$$

$$p = \left[\frac{(2-v)}{v}\right]^2$$

$$\lambda_1 = 0$$

$$\lambda_2 = -\frac{v}{2(2-v)}; v < 2$$

$$\lambda_3 = -\frac{2v}{(2-v)}; v < 2$$

First observe that inflation cannot be negative. To allow for deflation we must admit, that $v'^2 \neq 4p$, accordingly under $p<0$ we should have $v' = 2i\sqrt{|p|}$ and $v'^2 = -4p$. So if deflation prevails $p = -\left[\frac{(2-v)}{v}\right]^2$ holds.

Observe also that in order to attain convergence the initial velocity v should be less then 2. Two is the lowest possible number of circulations for equilibrium convergence in a monetary economy. The first circulation gives information about market imbalances and the second eventually eliminates them. If the economic agents planning horizon implies less then two circulations, the velocity accelerates at rate inversely proportional to initial velocity. If the horizon is longer then 2 circulations, the velocity declines.

We need however a more rigorous analysis of the relation between velocity and time horizon. First observe that $T > t_c = 1/v$. Where T is the time period chosen for the calculation of velocity, one year for example; t_c is the circulation period and equals the inverse of velocity. If we assume that the calculation and the circulation periods coincide, we can deduce:

(16) $\partial t_c / \partial t = t_c' = -(\partial v / \partial t)/v^2 = -v'/v^2 = -\frac{2(2-v)}{v^3}$

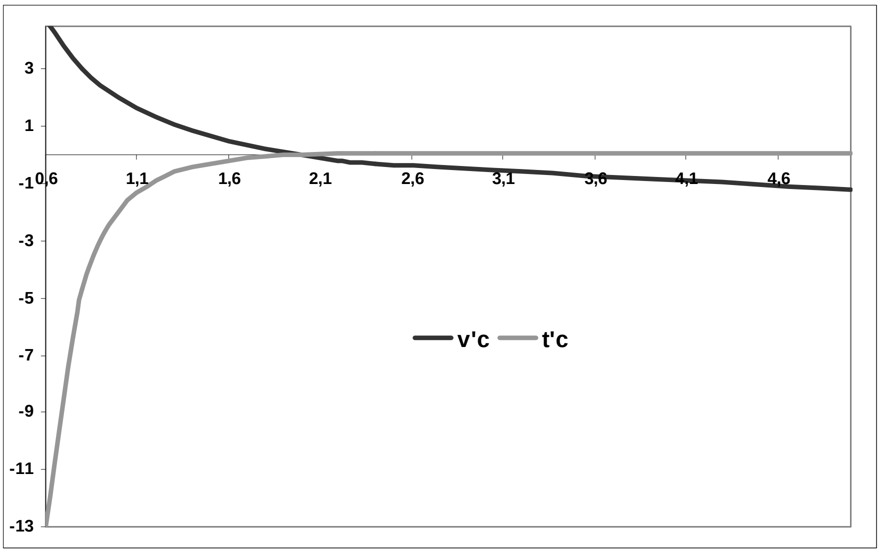
The figure 1 represents the relationship between the initial velocity, velocity acceleration and the speed of change of circulation period t_c'. The lower v (the higher the initial distance from equilibrium), the higher the velocity acceleration v'. The speed of change of circulation period is strongly negative. Velocity exceeding 2 rules out convergence but allows for the shift of equilibrium itself. For higher then 2 rate we observe slow velocity decline and even slower circulation time lengthening.

So we can reconstruct the following mechanism of short-term convergence and long term growth.

If some external shock affects the economy (velocity decelerates), the planning horizon (inverse of velocity) increases. From short term convergence point of view increased horizons signify higher distance from equilibrium.

If the planning horizon coincides with circulation time, then any shortening of horizon generates acceleration of velocity. When the economy is sufficiently close to equilibrium, the financial markets allow for coordination of future activities and the planning horizon quickly stabilizes. The velocity gradually increases to more then two circulations and the economy re-switches from convergence to long-term growth. The income velocity instigates decline. The long term planning horizon and circulation period slowly increase.

Figure 1



The re-switching may be helped by a swap to transitory deflation and back to moderate inflation. Such shift implies alteration of fast convergence and divergence, according to (15). So a transient deflation episode may be helpful if followed by velocity deceleration.

The circulation time t_c maybe viewed as some kind of internal time horizon proper to the economic system. The velocity measured in internal time is different compared to external independent time velocity. In particular, if the circulation time increases the velocity declines and vice-versa.

If we interpret velocity as internal time determined, it should be considered oscillating around 2, since if it exceeds this number the acceleration becomes negative and turns out to be positive in the opposite case. Any change of velocity and circulation time contradicts the initial duration structure of financial assets and liabilities. If current velocity exceeds the initial one more than 2 times, more than 50% of contracts must be rescheduled, so the system is destabilized and new institutional (contractual) structure is necessary. This is similar to re-contracting under tâtonnement process.

It must be added also, that interdependences between velocity and planning horizons have different meaning in short and long run. In the short run velocity acceleration and horizons shortening reflect equilibrium convergence, while in the long run velocity deceleration indicates institutional and technological shifts.

Circulation time requiring two circulations, or $t_c = \frac{1}{2}$ could be interpreted also as Lyapunov time (period throughout which the system is stable) since more then two circulations imply divergence.

We must state that if velocity measured in terms of circulation time (v_c) and velocity in fixed calculation period *T* (v_T) match, the first derivatives should also be equal, or $\partial v_c / \partial t = \partial v_T / \partial t$ and

$$\partial v_c / \partial t_c = [\partial v_c / \partial t]/[\partial t_c / \partial t] = [2(2 - v_c)/v_c]/[-2(2 - v_c)/v_c^{\ 3}] = -v_c^{\ 2} = -\frac{1}{t_c^2}.$$

Consequently increasing time horizon implies declining velocity in terms of circulation time. Note also that the velocity acceleration is always negative ($v'' < 0$).

The variables λ_2 *and* λ_3 can be interpreted as the “price” in terms of velocity acceleration (and consequently of additional monetary income) of keeping inflation under some threshold (λ_2) and keeping the money supply growth nonnegative (λ_3).

Both variables are negative, so monetary policy of keeping the money supply fixed and price change at low pace has economic cost in terms of negative contribution to growth.

Decreasing money supply can also be a solution to the problem of monetary policy, especially in case of strong initial monetary overhang. Declining money supply requires however modification of the Lagrangian system, namely the term after λ_3 must change the sign.

In addition, we must take into account that if contractionary monetary policy generates deflation, the economy will further diverge from equilibrium. Thus we should impose non-positive discriminate condition, namely $4p - v'^2 \geq 0$. So we obtain:

$$\Phi = v' + \lambda_1(v' - t) + \lambda_2(4p - v'^2) + \lambda_3(\dot{v} - \dot{y} - p)$$
$$\partial\Phi / \partial v' = 1 + \lambda_1 + 2\lambda_2 v' + \lambda_3(1/v - 1/2) = 0$$
$$\partial\Phi / \partial p = -4\lambda_2 - \lambda_3 = 4\lambda_2 + \lambda_3 = 0$$
$$\partial\Phi / \partial t = -\lambda_1 = 0$$
$$\lambda_2(4p - v'^2) = 0$$
$$\lambda_3(v'/v - v'/2 - p) = 0$$

The solutions are:

$$v' = \frac{2(2-v)}{v}$$
$$p = \left[\frac{(2-v)}{v}\right]^2$$
$$\lambda_2 = \frac{v}{2(2-v)}; v < 2$$
$$\lambda_3 = \frac{2v}{(2-v)}; v < 2$$

The difference between the non-negative and the non-positive money supply growth is that in the latter case the control on the money supply with relatively high inflation implies higher growth. Given that real money supply contraction and high inflation are in principle mutually exclusive in the long run, we must reckon that contractionary monetary policy could generate transient deflation episode before the economy returns to long term trajectory.

In the long run however, the interdependences change the sign. A low-inflation non-negative money supply policy becomes beneficial, while high inflation contractionary monetary policy is obviously counterproductive.

The fact that both overheating and underemployment of resources require velocity acceleration has long term consequences. It implies that long-term evolution should necessarily take the form of declining income velocity of money. The latter implies also that in the long run the economic agents should maximize $\left| -v' \right|$ and not v'.

We need to establish long-run relations between growth and money. The relationship between current NNP, consumption maximization, economic growth, production possibilities constraints and financial markets, is discussed in Appendix 2.

Let us suppose that the long term equilibrium growth rate is represented by the expression $y^{*\prime}/y^*$, where $y^* = y_0^* e^{r^* t}$, and the equilibrium rate itself equals r^*.

We presume also that in the long run the deviation between equilibrium and current NNP is driving the growth process.

This means that the causal relationships between the equilibrium and the effective NNP are reversed. In the long run the effective NNP perpetually deviates from the initial equilibrium and the equilibrium NNP converges to the effective NNP.

This is logically possible if we define the monetary economy as an instrument of long run economic coordination. The economic agents are coordinating long term plans via issuing financial instruments (claims on future cash flows). These financial instruments are quoted (exchanged for money) on the financial markets. This means that long run monetary aggregates dynamics must precede the real sector equilibrium shifts (broad money velocity deceleration). It is also clear that from long term point of view the broad money income velocity is the best possible indicator.

The latter reasoning applies only to expected or pre-planned developments. Unexpected shocks, on the contrary, should generate (short-run) equilibrium convergence via overall velocity acceleration.

We presume that in the long run the equilibrium NNP rate and the convergence rate are equal:

(17) $y^{*\prime}/y^* = \bar{y}'/y^*$

The equation (17) implies that economic agents forecast infinitely lasting positive equilibrium shifts. Assuming in addition that in the long run the discriminant of the equation (12) is zero, we can deduce from (17) the following equality:

(18) $y_0^* r^* e^{r^* t} = -\bar{y}_0 \frac{v_b'}{2} e^{-\frac{v_b'}{2} t}$

Furthermore we allow $y_0^* = \bar{y}_0$. We reformulate the expression of equilibrium divergence to $\bar{y}_0 = y_0 - y_0^*$. Since the initial equilibrium NNP level cannot be 0, the only possibility is that $y_0 = 2y_0^*$. In other words the long run convergence should start with current NNP exceeding the equilibrium. It also means that the financial sector should lead the real one.

The leading role simply reflects the fact that the long term investment plans coordination in a decentralized economy must take the form of contractual obligations (financial claims) and the activating of the obligations (issuing financial assets) must precede the fulfillment by definition.

Under these conditions, the equation (18) implies $r^* = -\frac{v_b'}{2}$. Clearly r^* can be positive if and only if the long run velocity of (broad) money declines ($v_b' < 0$).

The problems of short run equilibrium convergence and long run growth maybe analyzed also in terms of Lyapunov exponent (see the Appendix 1).

Finally, the effective growth rate is the half of the long run declining velocity of broad money, plus temporary velocity accelerations, generated by stochastic positive and negative external shocks.

5. Conclusions and Policy Implications

The paper is focused on the utility of money. The latter is defined as the ability of money to broaden the exchange (by eliminating the double coincidence of wants constraint). The (local) utility of money is measured by the speed of convergence of monetary income to equilibrium. This speed in turn is determined by the acceleration of income velocity of money.

In other words, the purpose to maximize monetary income is equivalent to the objective of maximizing the utility of money. In turn, maximizing monetary income via (modified) Roy's identity is equivalent to maximizing present and future consumption.

Since the utility of money is additive and the velocity integrates all the agents, we can relate the problem of utility maximization and the monetary policy. This allows us to discuss the problems of monetary policy directly in terms of utility maximization.

It follows from the present paper that in the short run velocity deceleration is the main sign of economic downturn, while acceleration is symptom of recovery. Whether authorities could affect velocity by an appropriate policy and stabilize economy is an open question.

The overall velocity of circulation cannot be considerer neither stable nor even predicable for it is the main instrument of external shocks adjustment.

Nevertheless, in the short run, moderate stabilization oriented monetary policy must prevail, even brief deflationary periods maybe acceptable. The latter feature is compatible with neoclassical-monetarist tradition.

In the long run the monetary policy should allow for some kind of lead for the monetary/financial parameters of the economy.

Another result is that we can expect the monetary/financial sector to be the usual generator of macroeconomic instability. Including financial sector in the integrated economy we cannot realistically assume gross substitutability between financial market volumes and prices and the real sector ones. Financial bubbles and depressions destabilize the whole economy.

It is also clear, that the monetary policy have different objectives and instruments in short and long run.

The general implication is that stable economic development requires financial markets regulation and supervision. In view of the fact that we have problems in terms of stability of equilibrium convergence, synchronization and short-termism, we need broader range of instruments, than simply CB interest rate policy. The conclusion about insufficiency of CB interest rate and the role of the so-called "velocity residuals" (i.e. unexpected velocity shocks affecting the economy) is empirically confirmed.[25] In addition to this, cyclically adjusted capital adequacy and obligatory reserve policies are indispensable.

The model can be transformed into non-homogenous equation representation and into stochastic differential equation version. The stochastic differential equation approach requires reformulation of the basic equations adding stochastic term.

The differential equation can also be reformulated in variable coefficients form. Additional analysis is required for better understanding of the re-switching from short-term convergence to long term growth.

Since the velocity is erratic and since the circulation of money always implies uncertainty, the velocity of money is fundamentally related to information-theoretic entropy rate.

This line of reasoning allows for deeper understanding of the role of money. The money is actually a veil, but veil conveying information. Money, on the other hand, must be neutral and the nominal quantity of money should not affect the real economy. The only way these restrictive requirements to be met, while allowing for some effective role of money, is to assume variability of the income velocity.

In fact the velocity of money is the only money-related variable which, at least in the long run, can be considered to be a zero homogeneity function of the price level. By the same reason it is the single candidate for a pecuniary variable, affecting the real economy.

[25] Reynard, S. Maintaining Low Inflation: Money, Interest Rate and Policy Stance. – ECB Working Paper Series, N 756, May 2007.

Given the fact that present (2009) global economic crisis was preceded by broad velocity deceleration, the former may be attributed to an extreme form of financial myopia. In particular, from Lyapunov exponent interpretation, we now that velocity, higher than certain threshold[26], destabilizes the system.

The velocity accelerations discussed in this paper may be difficult to observe because they may require very small periods (one month for example) for which usually GDP or other aggregate income measures are not computed.

Nevertheless the empirical data is generally compatible with the model-deduced conclusions, what can be seen from the empirical research the papers of Razzak[27], McClam[28], Zholood[29] and others.

Our conclusions also broadly corresponds to Bordo and Jonung's reasoning that "no single theory can explain both the secular decline and rise of velocity".[30] It also is consistent with the distinction between secular and cyclical trends by Friedman[31] and the distinction between circuit velocity and real velocity by Keynes[32].

The big unanswered in this paper problem is the Bordo and Jonung's hypothesis about the role of institutional factors in explaining long term deceleration of broad income velocity of money. This question needs additional attention.

On the other hand, the other Bordo and Jonung thesis about technological innovations as explanation of acceleration periods of both broad and narrow income velocity of money is compatible with our model since our framework allows for different types of external shocks.

In spite of Wicksell's statement that "the purely physical conditions under which money can be paid and transported set a definite limit to the magnitude of the velocity of circulation"[33], it is clear that the modern information technologies allow for income velocity of circulation so high, that it could not effectively bind the speed

[26] In this case the velocity must be calculated in circulation time.
[27] Razzak, W. A. Money in the Era of Inflation Targeting. Reserve Bank of New Zealand, DP 2001/02.
[28] McClam, D. W. US Monetary Aggregates, Income Velocity and the Euro-Dollar Market. – BIS Economic Papers, N 2, April 1980.
[29] Zholood, O. Volatility of Velocity in Transitional Economies: Case of Ukraine. The National University of Kiev-Mohyla Academia, 2001. Appendix 1
[30] Bordo, M. D., L. Jonung. The Global Velocity Curve 1952-1982. National Bureau of Economic Research, Working Paper N 2074, 1986.
[31] Friedman, M. The Demand for Money: Some Theoretical and Empirical Results. – Journal of Political Economy, Vol 67, August 1959, p. 327-351.
[32] Keynes, J. M. The General Theory of Employment, Interest, and Money. London: Macmillan, 1936.
[33] Wicksell, K. Interest and prices. Translated by Kahn, R.F. London: Macmillan, 1936, 1898.

of transactions[34] and that the effective velocity is endogenous (determined by economic forces).

Another topic related to income velocity of money is the impact of interest rate. This relation was discussed by prominent authors, like Friedman (1959). High interest rates accelerate velocity and vice versa. In our model, the interest rate is settled on the real market as price, balancing supply and demand of loanable funds. This price is one of the factors, determining the real quantity of money in circulation. Since economic agents are maximizing the utility of money via velocity and taken into account that they are supposed to lend and borrow freely, we should observe connection between interest rate and velocity acceleration. In particular velocity should accelerate or decelerate at absolute rate matching prevailing interest rate.

The conclusions drown in Section 4 about the negative relationship between restrictive monetary policy and velocity are theoretically and empirically confirmed.[35]

Appendix 1

The Lyapunov exponent is mathematical technique used to study the dynamical systems dependence on the initial conditions. The Lyapunov exponent (for one dimensional case) is formulated as (see Gaspard: 2004):

$$\lambda = \frac{\lim}{t \to \infty} \frac{1}{t} \ln \frac{|\partial x(t)|}{|\partial x(0)|}$$

Where $\partial x(0)$ is the initial condition. In our case the converging variable is $\overline{y}$. We assume for simplicity that the equilibrium convergence is described by equation (13).

So the Lyapunov exponent takes a simple form:

$$\lambda = \frac{\lim}{t \to \infty} \frac{1}{t} \ln \frac{|\partial \overline{y}(t)|}{|\partial \overline{y}(0)|} = \frac{\lim}{t \to \infty} \frac{1}{t} \ln \left| e^{-\frac{v'}{2}t} \right| = -\frac{v'}{2}$$

In view of the Lagrangian solutions we obtain:

[34] Wicksell's postulate is obviously true but the maximum physical velocity may be attained if and only if the precautionary motive shrinks to zero. Nevertheless, if we neglect reserves, the quantity of money "in motion" may be assumed to approach this maximum velocity.

[35] Padrini, F. Efficiency of the Payment System, Velocity of Circulation of Money, And Financial Markets. Ministry of the Treasury, Italy, November 1996.

$$\lambda = -\frac{v'}{2} = \frac{(v-2)}{v}$$

Where *v* is the initial velocity. It ensues that for a set of initial conditions $v < 2$ the system is stable while for $v > 2$ the system diverges and is strongly contingent upon initial conditions.

Since v' is an decreasing function of the time period (here we assume that $T = t_c$ *and* $v_c = v_T$) it follows that he system converges for increasing velocities up to $v_c = 2$, while circulation time declines from infinity to ½. Higher then 2 velocity implies divergence. As far as the system depends on initial velocity definition, it is clear that periodically the system should re-switch to longer basic circulation periods in order to avoid destabilization. This can be interpreted as a necessary institutional change implying longer circulation periods. In other words, the system must be periodically redefined on the basis of more complicated and respectively longer circulation periods. In the long run only institutional and technological change can guarantee growth (population related growth excluded).

The Lyapunov exponent can also be interpreted in terms of Kolmogorov-Sinai entropy. If the average entropy produced by the change of initial conditions is positive the system dissipates. If the entropy rate is negative the system converges to a more ordered state.

In the multidimensional case more complicated trajectories, including chaotic behavior are and limit cycles are admissible. The Lyapunov exponent can take even more complicated forms in the case of equations (23) and (24).

Another conclusion is that the system is self-stabilizing if velocity increases. If compound velocity (broad income velocity) declines the system is unstable and must be regulated. Alternatively it should go through chaotic restructuring via spontaneous velocity acceleration.

Appendix 2

The exposition is based on Weitzman (Weitzman, 1976, ibid) and on Sterken.[36] We assume that all sources of economic growth are attributed to capital, broadly defined. The consumption may be represented by any cardinal utility function or index number with fixed weights. There are n capital goods, the stock of capital in

[36] Sterken, E. Lecture 3: Theory of Optimal Control, Institute of Economics and Econometrics. Faculteit der Economische Wetenschappen Rijksuniversiteit Groningen, March 2007.

time t is denoted as $K_i(t)$. Net investment flow is $I_i(t) \equiv dK_i/dt$. The production possibility set is expressed as $S(K(t))$. Any consumption-investment pair may be produced if and only if $(C, I) \ni S(K(t))$. The prices p_i of the capital goods are deflated by the consumption price of unity, i.e. the consumption is used as numéraire. The real net national product function is defined as follows $Y(K, p) \equiv \max[C + pI]$. Any feasible trajectory satisfies the conditions: $\{C(t), \frac{dK}{dt}(t)\} \ni S(K(t))$ and $K(0) = K_0$.

A competitive trajectory meets also the condition capital goods prices to exist and be equal to the marginal rate of substitution, given the rate of return r. These conditions are defined as: $Y\{K(t), p(t)\} = C(t) + p(t)\frac{dK}{dt}(t)$ and $\frac{\partial Y}{\partial K_i} = rp_i(t) - \frac{dp_i}{dt}(t); i = 1,2..,n$. The optimal control problem is formulated as:

(1) Maximize $\int_0^\infty C(t)e^{-rt}dt$

(2) Subject to $\{C(t), \frac{dK}{dy}(t)\} \ni S(K(t))$

(3) $K(0) = K_0$

This reduces to the differential equation $\frac{dY}{dt}(t) = r(Y(t) - C(t))$ with solution $Y(t) = r\int_t^\infty C(s)e^{-r(s-t)}ds$.

In a monetary economy with augmented Clower constraint we can replace capital goods with financial instruments. Since the accumulation of physical capital must be financed, we assume that $pK = M$, where M is the broad money aggregate. We suppose also that every capital good is financed via special financial instrument so that to any pair p_iK_i corresponds pair $\tilde{p}_i\tilde{M}_i$ and $p_iK_i = \tilde{p}_i\tilde{M}_i = M_i$ where

$\widetilde{M}_i$ is the net present value of the instrument. In such a way we can replace everywhere K_i by $\widetilde{M}_i$ and p_i by $\widetilde{p}_i$ where $\widetilde{p}_i = 1 + r_i$.

So we can formulate the problem in the following way.

Maximize $\int_0^\infty C(t)e^{-rt}$

Subject to $C(t) + M^{'} = F(M_t)$

Where M is the broad money what in our case means total market capitalization. The consumption is continuous twice differentiable function of time.

We can rewrite the constraint as $M^{'} = F(M_t) - C(t)$. Define the Hamiltonian as $H(t) = C(t)e^{-rt} + \mu(t)[F(M_t) - C(t)]$. In this case $\mu(t)$ is a dynamic Lagrange parameter. To get a current value Hamiltonian we introduce $\lambda(t) = \mu(t)e^{rt}$. We can write now $H = [C(t) - \lambda(t)[F(M_t) - C(t)]]e^{-rt}$.

We can formulate the following solution conditions:

Condition 1: $H_c^{'} = 0$

Condition 2: $\mu^{'}(t) = -H_M^{'}$

Condition 3: $\lim \mu(t)F(M_t) = 0$ if $t \to 0$.

From Condition 2 we can obtain $\lambda(t)^{'} = \lambda(t)[r - F(M_t)]$. The coefficient $\lambda(t)$ can be interpreted as the utility of infinitesimal increase of M. Thus we can rewrite $\lambda(t)^{'} = \lambda(t)[r + F(M_t)] = C(t)^{'} = \left(\frac{C(t)^{'}}{-C(t)^{''}}\right)[r - F(M_t)] =$ $\delta(C(t))[F(M_t) - r]$ where $1/\delta(C(t)) = -C(t)^{''}/C(t)^{'}$.

We can further write:

$$C(t)' = \delta(C(t))[F(M_t) - r]$$

$$M(t)' = F(M_t) - r$$

In the steady state we have $C(t)' = M(t)' = 0$. Consequently $F(M_t)' = r$ and $C(M^*) = F(M^*)$.

To study the behavior of the model around the steady state we can linearize the system:

$$M(T)' = -(C(t) - C^*) + F(M_t)'(M_t - M^*)$$

$$C(t)' = -\beta(M_t - M^*); -\beta = \frac{C(t)'}{(M_t - M^*)}$$

Combining both above equations we obtain:

$$M_t'' - rM_t' - \beta(M_t - M^*) = 0$$

This is a second order differential equation, similar to equation (20). Remind that we are studying linear approximation system. So we can write $M_t'' = (M_t - M^*)'' = q\bar{y}''$, $M_t' = (M_t - M^*)' = q\bar{y}'$ and $M_t - M^* = q\bar{y}$. Observe also that with free lending and borrowing we have $v' = r$. This means that under disequilibrium in a monetary economy with augmented Clower constraint, the economic agents may choose between increasing income via accelerating velocity or via obtaining interest on financial instruments (deposits).

Note also that since we used consumption price index as numéraire we can interpret inflation rate as shift of the relative price of financial instruments (investment goods) in terms of consumption price index. So we can define coefficient β as inflation or more precisely $\beta = \frac{P'}{P}$. It is natural to assume that relation between consumption adjustment and financial market adjustment depends on relative price dynamics.

In such a case we can derive:

$$M_t'' - rM_t' - \beta(M_t - M^*) = q\bar{y}'' - qv'\bar{y}' - qp\bar{y} = \bar{y}'' - v'\bar{y}' - p\bar{y} = 0$$

In other words the conditions, derived as maximization of the utility of money at micro (equation (20)) and macro level coincide with the conditions in terms of long term consumption maximization.