Luben T. Toshkov[1]
Tatyana S. Madjarova[2]
Tzvetan Ignatov[3]

LORENZ CURVE AND THE MEASUREMENT OF LOW, MIDDLE AND HIGH STRATA OF INCOMES

The authors suggest a new method to determinate the limits of the three basic income groups. The Lorenz curve is split of three parts. The middle part corresponds to the middle stratum of incomes. The projections of this part on both axes are equal segments. The empirical Lorenz curve is transformed in new one with the same Gini coefficient. The transformed curve is composed by three segments with "clearly expressed the three strata of incomes". The proposed model is used with some statistical data for the incomes and the wealth of the households in Bulgaria and Germany. The calculations are made with iterative methods in Microsoft Excel.
JEL: O15,C46, C51

1. Introduction

In this paper we use the concept "strata" with goal to classify the population (persons or households) on three groups only by the criterion the incomes (or the wealth). This criterion is not sufficient for determination of the social-political concepts "class", "poor" and "rich". For that reason are not identical the pair notions: stratum and class, low stratum and the poor, middle stratum and middle class, high stratum and the rich. We will not consider the problems with the poor, the middle class and the rich.

The purpose of this research is to construct the three strata using Lorenz curve. In practice these three strata are defined by different ways.

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[1] *Luben T. Toshkov is from Sofia University "St. Kliment Ohridski", Faculty of Economics and Business Administration, e-mail: toshkov@feb.uni-sofia.bg. He has worked on secion 1 and 3.*
[2] *Tatyana S. Madjarova is from Naval Academy "Nikola Vaptsarov", Faculty of Engineering, e-mail: madjarova@mail.com. She has worked on section 2 and the calculation in Table 1.*
[3] *Tzvetan Ignatov is from Sofia University "St. Kliment Ohridski", Faculty of Economics and Business Administration, e-mail: ignatov@feb.uni-sofia.bg. He has worked on section 2 and the calculation in Table 1.*

Frequently in the scientific literature and in the political practice authors use as definition of the three strata some different points around the statistical estimates of median or average of incomes. For example in Germany the middle stratum was defined as the population with:

- income (around the median) between 75-125% (Peter Berger, 2005));

- income (around the median) between 70-150% (Markus Grabka, 2008));

- income (around the median) between 75-150% (Lebenslagen in Deutschland, 2008); etc.

In some cases the three strata are defined by choosing the absolute valuations of incomes. Some examples from Bulgarian authors:

- consulting company Credit Centre in Bulgaria accept that in 2009 year the middle strata includes everyone with income between 750-1500 BGN monthly (investor.bg, 06.08.2009; 12:09);

- researchers from the Institute for market economics in Bulgaria accept that in 2005 the middle strata had between 700-900 BGN net income monthly (Стандарт, 16.06.2005);

- other author analyzes the inequality of the wage in Bulgaria in the period 2000-2006 as he uses three levels: low, equal of the official governmental minimal wage (150 BGN monthly in 2005); middle, equal to twice of the low level (300 BGN) and high, equal to twice of the middle level (600 BGN) (Димитър Нинов, 2008); etc.

The above stated and other similar examples of subjective, arbitrary determination of the limits and the levels show that it is necessary to find a precise quantitative method for defining the three strata. In our opinion the basic methodological defect of the numerous made experiments for stratification is in this, that they are confined in the so-called "Pareto distribution". Many authors reduce the problem for stratification in finding the relative parts of the persons (or the households) disregarding the relative parts of the same persons (or the households) in their total income. The attaching examples in section 3 of our paper show on principle the opposite directions of the two relative parts in this case, that we recede from the unique point of the Lorenz curve, in which the two relative parts are absolutely equal, in direction to the lowest income groups, or in direction to the highest income groups.

The Lorenz curve gives a very good possibility for overcoming of the outstanding methodological defect. By the decision of this problem with the help of the mathematics in section 2, we proceed from the basic idea that the middle stratum is presented with this segment of the Lorenz curve, in which the relative parts of

incomes and population are equal, i.e. the only segment, which has equal projection of the absciss and the ordinate.

2. Construction of three basic strata using Lorenz curves.

Let us recall some properties of Lorenz curve $L(x)$.

- Economical interpretation of the curve $L(x)$ is the x part of the low income population has the $L(x)$ part in the total income.

- The Lorenz function $L(x)$ is defined on [0,1]; $L(0)=0$ and $L(1)=1$.

- The Lorenz curve $L(x)$ is convex and increasing function on unity interval.

The fundamental idea adduced in the end of section 1 is illustrated in one standard graphic of the Lorenz curve (see Graphic 1). As the graphic shows the analytical expression of our idea is the following starting equation:

$$v - u = L(v) - L(u).$$ (1)

How we can prove this equation?

If G is a Gini coefficient of the Lorenz curve $L(x)$ we construct an isosceles trapezium ABCD with the area coincides to the area of the domain between the main diagonal AB and the Lorenz curve $L(x)$. That means

$$S_{ABCD} = \frac{G}{2}.$$ (2)

Here we propose to consider the polygon ABCD as a Lorenz curve $\tilde{L}(x)$. Why do we choose this polygon? The polygon ABCD is also a Lorenz curve with the clear three strata and $\tilde{L}(x)$ is approximation of the $L(x)$ with the same Gini coefficient, i.e.
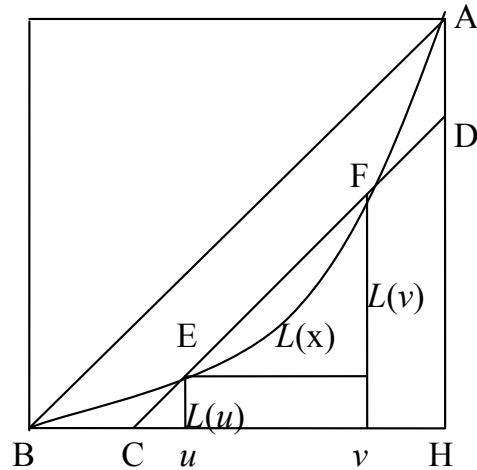
$$G_{L(x)} = G_{\tilde{L}(x)}.$$

Let points E and F be with co-ordinates E(u, $L(u)$) and F(v, $L(v)$).

Then we can express BC and CD in such a way

$$BC = u - L(u), \; CD = CH\sqrt{2} = (BH - BC)\sqrt{2} = \left[1 - (u - L(u))\right]\sqrt{2}.$$

Graphic 1

For the area of the isosceles trapezium can be receive

$$S_{ABCD} = \frac{AB + CD}{2} \cdot \frac{BC}{\sqrt{2}},$$

$$S_{ABCD} = \frac{\sqrt{2} + \sqrt{2}\left[1 - \left(u - L(u)\right)\right]}{2} \cdot \frac{u - L(u)}{\sqrt{2}}. \qquad (3)$$

From (2) and (3) we find

$$\frac{G}{2} = \frac{\sqrt{2} + \sqrt{2}\left[1 - \left(u - L(u)\right)\right]}{2} \cdot \frac{u - L(u)}{\sqrt{2}}.$$

After the substitution $r = u - L(u)$ upper equation can be given as

$$\frac{G}{2} = \frac{\sqrt{2} + \sqrt{2}\left[1 - r\right]}{2} \cdot \frac{r}{\sqrt{2}},$$

$$G = r + r(1 - r),$$

$$r^2 - 2r + G = 0, \text{ from where}$$

$$r_{1,2} = 1 \pm \sqrt{1-G} .$$

From the properties of the Lorenz function is known that $r = u - L(u)$ takes values between 0 and 1. Consequently

$$r = 1 - \sqrt{1-G} ,$$

i.e. $\qquad u - L(u) = 1 - \sqrt{1-G} .$ $\hspace{3cm}$ (4)

Thereby we receive the system of equations (1) and (4)

$$\left|\begin{array}{l} v - u = L(v) - L(u) \\ u - L(u) = 1 - \sqrt{1-G} \end{array}\right. ,$$

$$\left|\begin{array}{l} v - u = L(v) - L(u) \\ \quad u - L(u) = r \end{array}\right. .$$

We use an iterative procedure to find solutions of the system. From the recurrent equation $y_{n+1} = L(y_n) + r$ we obtain consequently the increasing sequence of the values $y_1, y_2, \ldots, y_n, \ldots$. if we start with $0 < y_1 < u$. If the beginning is the point z_1 so that $u < z_1 < v$ we obtain the decreasing sequence $z_1, z_2, \ldots, z_n, \ldots$ the both sequences tend to u. The value u is the solution of the equation: $u - L(u) = r$. Let us show these reasoning graphically (see Graphic 2).

Graphic 2

$0 \; y_{-1} \; y_n \; u \; z_n \; \ldots z_2 \; z_1$
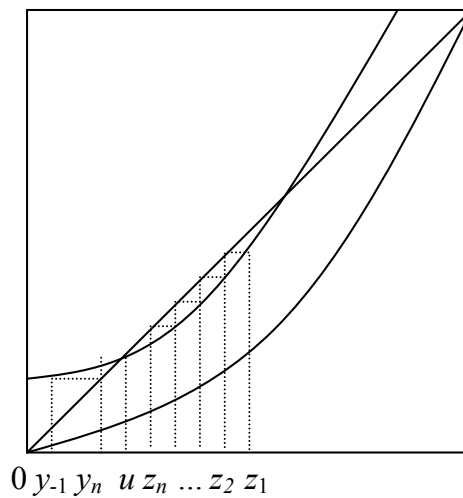
By analogue we find the right point of the interval of the middle stratum.

3. Applications

Using the consideration method we calculate the indicators of the three basic strata on some sources of data for Bulgaria and Germany. The results of the calculations with iterative methods in Microsoft Excel are given on the table 1.

Table 1

State; period and empirical coefficient of Gini (G)	Indicators of the strata	Low Strata	Middle Strata	High Strata
Bulgaria;1963 -1991; (G = 0,21)*	Share of persons (m) – % Share of gross income (n) – % Coefficient of stratification – (n:m)	27.00 16.00 0.59	56.10 56.10 1.00	16.90 27.90 1.65
Bulgaria;1992 -2007 (G = 0, 33)*	Share of persons (m) – % Share of gross income (n) – % Coefficient of stratification – (n:m)	33.60 15.80 0.47	53.30 53.30 1.00	13.10 30.90 2.36
Germany (West Germany) 2003 (G = 0,34)**	Share of households (m) – % Share of net income (n) – % Coefficient of stratification – (n:m)	31.50 12.70 0.40	52.30 52.30 1.00	16.20 34.90 2.15
Germany; 2003 (G = 0, 675)***	Share of households (m) – % Share of net wealth (n) – % Coefficient of stratification – (n:m)	44.30 2.30 0.05	40.80 40.80 1.00	14.90 56.90 3.82
Hypothetical example; (G = 0,855)	Share of households (m) – % Share of net wealth (n) – % Coefficient of stratification – (n:m)	64.70 2.70 0.04	27.60 27.60 1.00	7.70 69.70 9.05

* Source: Our calculations on basis of the official annual governmental statistics "Household budgets in the republic of Bulgaria", National statistical institute of Bulgaria, Sofia. For the period 1963-1991 there is a published information about inequality only for 8 years, and for the period 1992-2007 – there is for all years.
** Source: Statistisches Bundesamt (2007).
*** Source: Lebenslagen in Deutschland (2005).

We use Table 1 not to compare Bulgaria and Germany or to analyze of the inequality within them. We select the examples by criterion "grow of the coefficient of Gini (G)", which is the basic parameter for the decision of the starting equation (1). The existed statistical data for the inequality of the income of the households in both states don`t give a considerable differences (even between the two periods in Bulgaria with two divergence social-economical systems!). For that reason in addition we use one more statistical example for the inequality of the net wealth in Germany (G = 0,675) and also one made up hypothetical example for great inequality (G = 0,855).

As indicators of the three strata we use:

- the relative parts of the possessors of the income (or the wealth) in their total number, with indication m;

- the relative parts in the total income (or the total wealth) of the same possessors, with indication n;

- the ratio n:m, which presents the grade of the equivalent of the above stated relative parts.

The ratio n:m is the indicator of first importance by the stratification. For that reason we designate this indicator as a coefficient of stratification. By n:m = 1 is realized the starting equation (1), consequently in this way is determined the middle stratum. The low stratum is defined by n:m < 1, and the high stratum – respective by n:m > 1.

What does the development of the value of the indicators shows for every one stratum with reference to grow of the coefficient G in Table 1? The most important conclusions are the following:

First, the middle stratum is reducing but disproportionately of grow of the total inequality, i.e. this stratum shows a relative stability in comparison with other strata. While coefficient G is growing fourfold (from 0,21 to 0,855) the middle stratum is reducing approximately twice. Grow of G reflects strongly on the two end strata.

Second, the first two indicators of the low stratum develop contrary. The relative part of the persons (or the households) is growing, but their relative part in the total income (or the total wealth) is reducing. As a result of this the ratio n : m has an increasingly small value (approximate to zero).

Three, the first two indicators of the high stratum develop similarly contrary but in the opposite direction, i.e. the relative part of the persons (or the households) is reducing, but their relative part in the total income (or total wealth) is growing. The ratio n : m has increasingly big value over 1,00.

The above stated results present our method as a reliable for the objective verification of the three traditional strata of the population by incomes and by wealth. This method is practicable in the social policy, for example: in the taxation of the income and the wealth; in the social insurance; in the social assistance; etc.

4. References

[1] Bach, S.; Corneo, G. and Steiner, V. (2009). From bottom to top: the entire income distribution in Germany, 1992-2003.The review of income and wealth, series 55, 2, p.303-330.

[2] Berger, Peter (2005). Deutsche Ungleichheiten – eine Skizze.Aus Politik und Zeitgeschichte, 28-29, s.14.

[3] Dorfman, R. (1979).A formula for the Gini coefficient. Review of economics and statistics, 61, 146-149.

[4] Fellman, I. (1976).The effect of transformation on Lorenz curves. Econometrica, 44, 823-824.

[5] Grabka, Markus; Joachim Frick (2008). Schrumpfende Mittelschicht – Anzeichen einer dauerhaften Polarisierung der verfuegbaren Einkommen?. Wochenbericht des DIW,Berlin, 10, 102-103.

[6] Hagerbaumer, James B. (1977). The Gini concentration ratio and the minor concentration ratio: a two-parameter index of inequality. Review of economics and statistics, 59, 377-379.

[7] Lebenslagen in Deutschland (2005).Der 2. Armuts- und Reichtumsbericht der Bundesregierung,36-37, (http://www.bmas.de/portal/10070/lebenslagen__in__deutschland__der__2__a rmuts__und__reichtumsbericht__der__bundesregierung.html)

[8] Lebenslagen in Deutschland (2008).Der 3.Armuts- und Reichtumsbericht der Bundesregeirung,18,(http://www.bmas.de/portal/26896/lebenslagen__in__deut schland__der__3__armuts__und__reichtumsbericht__der__bundesregierung.ht ml)

[9] Statistisches Bundesamt (2007): Einkommensverteilung in Deutschland 2003, Fachserie 15 EVS, Heft 6, 37.

[10] Нинов, Димитър (2008): Социални цели и последици на бюджета и икономическите политики в България през 2000-2006 година, София, КНСБ, фондация Фридрих Еберт (Friedrich Ebert Stiftung), p.82.