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EQUILIBRIUM MODEL IN A MONETARY ECONOMY

The paper consists of three parts. In the first part a monetary economy, based on credit money, is introduced. All financial instruments are considered as substitutes for money. In the second part the possibility of equilibrium convergence, under certain conditions, is studied. The conclusion is that that the equilibrium convergence requires monetary velocity acceleration. The velocity of money is introduced as complex variable derived as solution of a matrix equation and implying the existence of closed money circulation cycles. The third part is dedicated to interplay between the real and the financial sectors under the process of equilibrium convergence and economic stabilization. Keynesian, monetarist and real business cycle type of stabilization are studied. The conclusion about the possibility of destabilizing role of financial sector is drawn.

JEL: E19

Monetary Economy with Credit Money

The monetary economy eliminates the double coincidence of wants constraint (Jevons, 1875), thus broadening the range of the possible exchanges. On the other hand, since the counterparties' bilateral exchange objectives do not match each other, all information is not revealed and additional interactions are necessary to attain general equilibrium. The medium of exchange, transmitting signals from one agent to another, becomes an indispensable instrument of removing market misalignments. Intuitively, the speed of such transmission is crucial in terms of equilibrium convergence.

To consider the problem, we come up with an economy, where money circulates equally well under disequilibrium and equilibrium, eliminating Jevons' double coincidence of wants (DCW) constraint.

For this purpose we assume monetary economy with Clower rule- "money buys goods and goods buy money; but goods do not buy goods" (Clower, 1967). We impose however additional constraint. Money buys financial instruments (including deposits) and financial instruments buy money, but financial instruments never buy goods or other financial instruments.[1]

Such augmented Clower constraint shrinks all financial instruments to different types of money substitutes and allows for construction of stylized financial system reduced to banking system. In addition, these rules imply that any change in the real economy has monetary dimensions. To describe such an economy we need further elaborations.

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[1] It means, for example, that financial instruments such as bills of exchange are either not used or are immediately discounted with the banking system.

The elimination of the DCW rule is possible only as simultaneous removal of interpersonal and temporal constraints on exchange. That is, any non-DCW trade rule implies saving- one of the participants, namely the seller, does not obtain immediately the goods and services he or she needs, but at least for the interim he or she “saves” the income. The duration of saving may be very short.

The other side in exchange, the buyer, either uses its prior savings, or obtains financing from a third part- the banking (financial) system. Prior savings may exist only if institutions issuing money are available. So at the origin of any monetary exchange we need implicitly three types of agents- seller (also saver-lender), buyer (also borrower-spender) and financial intermediary (bank).

This mechanism entails also combination of functions of money as store of value, medium of exchange and numéraire and allows for the elimination of the DCW rule via the mechanism of generating saving and borrowing. This allows us to put up a simple economy with banking system supplied money.

Our approach of modeling money[2] and credit differs from both exchange credit and asset capital approaches (see for details in Gillman, Siklos and Silver, 1997).[3]

We unify money and credit since both diverge only in details (duration) while sharing the same origin (deposits and loans are based on the circulation of currency via the multiplier). The other financial assets are also derived from money as far as they must be swapped for cash to fulfill their functions.

Let’s define a general equilibrium system with *n* goods and financial instruments[4], money being one of the financial instruments. Money market is supposed to be an exchange of (electronic) cash against short-term financial instruments (demand deposits).

The equilibrium vector product is defined as $\hat{p} \cdot \hat{x}$ where $\hat{p}$ and $\hat{x}$ are the equilibrium values of prices and quantities ($\hat{p} \cdot \hat{x} \geq p \cdot x$) of all *n* real goods and financial instruments. Prices are defined as the number of monetary units, necessary for the purchase of one unit of the respective goods.

The monetary income, generated by the economic units, depends on prices (quantity of money per unit of commodities) and the velocity of circulation. So the income can be expressed as $(p \cdot i) \cdot v$, where *p*, *i* and *v* are the price and unit vectors (units definition will be explained later) and the velocity of money respectively. Since, by definition, prices are equal to quantity of money per unit of respective good, we have:

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2 We assume the creation of money ex nihilo in accordance with the so called circuit theory of money see for details Panagopoulos & Spiliotis, 2005.

3 According the above mentioned authors the credit is the closest substitute for money. We however share the view of the credit nature of money itself.

4 By equilibrium system of n markets we understand not only equality between supply and demand but also optimality in the sense that vector product of the optimal quantities and prices is greater or equal than the product of any other combination of prices and quantities.

(1) $$p \cdot i = \sum_{i=1}^{n} p_i = p_1 + p_2 + \ldots p_n = P = \sum_{i=1}^{n} m_i = m_1 + m_2 + \ldots m_n = M^0 ,$$

Where $m_1, m_2 \ldots m_n$ are the numbers of monetary units per piece of respective goods or instrument and $p_1 = m_1, p_2 = m_2, \ldots p_n = m_n$; M^0 and *P* are the quantity of money (monetary base) and price level respectively.

We assume further that the velocity of money is the same for all economic agents. This is a strong assumption for it requires all economic units and markets to be interconnected and synchronized via the monetary flows. This supposition will also be studied later.

We bring in a simple banking system. The Central bank is issuing money backed by Government and commercial bank assets (reserves). Banks are creating deposit money and supplying loans. Money in circulation exists only in electronic form as current accounts and deposits. Initially money is issued via lending to buyers in order to finance purchases. Money emerges as buyers' liability and sellers' asset.

This is in contrast with the traditional Cash in Advance (CIA) approach separating currency and lending. We argue that money is essentially a credit based instrument[5]. Cash in terms of commercial banks reserves with the central bank is equivalent to zero-coupon bond, yielding return from money related services.

There is no substantial difference between deposit money and the other financial instruments. If bonds, for example, are used to finance corporate investment in fixed capital or government spending, they should be sold for money on the financial market. The (electronic) cash obtained by the issuer will be returned to circulation (by purchasing investment goods), as in the case of the money multiplier. So we include in the broad money aggregate all financial instruments.

A banking system consisting only of a Central bank and commercial banks, issuing electronic deposit money and supplying loans, is a good stylized financial system, possessing all substantial properties, necessary for our analysis.

Further, every economic agent is assumed to supply goods and services using its initial resources' endowments. The agents however can sell not only goods, but also financial instruments to cover their deficit positions. The supply (production) of goods implies purchases of inputs from the other economic agents. Inputs include individual consumption. Supplies are exchanged for money and money is used to by inputs and financial instruments. The participants in the exchange are supposed to maximize utility.

[5] Even the central bank issued fiat currency (notes and coins in circulation) is always CB liability, usually backed by government debt. The latter can be viewed as commitment to supply gratis public goods in the future.

These initial purchases generate indebtedness to the banks, on the one part, and on the other part - income, stored as current accounts and deposits by buyers. Subsequently the agents can buy goods using their accounts or via borrowing from banking system (selling financial instruments). The proceedings from sales go either for credit repayment or for accumulation of deposits (purchase of financial instruments).

We assume also that the (broad) quantity of money in circulation equals the net present value of the additional future expected income (expenditure) of final lenders and debtors, free of transaction costs.

According to the thesis of Bhattacharaya and Thakor (Bhattacharaya and Thakor, 1993) and Broecker (Broecker, 1990) we can view the banks as coalitions of individual lenders and borrowers exploiting economies of scale and scope in transaction technology. So borrowers sell and lenders buy financial instruments via banks. The banks are intermediaries, selling also financial services, belonging to the aggregate supply and demand.

The banks indirectly guarantee the forthcoming supply of goods by controlling loans disbursement. Consequently, the financial markets belong to the implicit futures segment of the real economy. We consider the lending-borrowing activity as additional goods (financial instruments), whose price affect all the other markets.

To describe the network of payments and supplies of goods and services we have prepared a table, similar to matrix, summarizing financial flows between main sectors and markets in an open economy (see for example Gandolfo, 1978).

The rows of our table of payments summarize the market decomposition of payments and supplies (in this case n markets), while the columns reflect the transactions and the balances of the economic agents (*m* agents).

The first row summarizes the endowments of all *m* agents ($\sum_{i=1}^{n} p_i x_{ij} + m_j^0$). The endowments consist of real goods and financial instruments (quantities, multiplied by prices; the prices of the financial instruments are numbers, equalizing the face value of the instrument and its market price[6]) plus some cash m_j^0. Monetary base is included in the individual market equations, but is separated in endowments, because of its particular role in the process of income generation.

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[6] Suppose the face value of the instrument (deposit) is x and its period is one year. Its' net present value (not including interest) is $x_p = \frac{x}{(1+r)}$. The price p is $(1+r)$. The product $px_p = x$. So we can define the broad money as product of vectors of prices and NPV-es. When the interest is effectively calculated its added to the deposit amount and x increases to x_1.

Table of Payments

Agents Markets	1 $\sum_{i=1}^{n} p_i x_{i1} + m_1^0$	2 $\sum_{i=1}^{n} p_i x_{i2} + m_2^0$	…j $\sum_{i=1}^{n} p_i x_{ij} + m_j^0$	N $\sum_{i=1}^{n} p_i x_{im} + m_m^0$	$\sum_{i=1}^{n}\sum_{j=1}^{m}(p_i x_{ij} + m_j^0)$
1 $F_1; M_1^0 > \sum_{j=1}^{m} m_{1j}^0$	$\Delta m_{11}^0 = -p_1 x_{11}$	$\Delta m_{12}^0 = -p_1 x_{12}$	$\cdot\cdot\ \Delta m_{1j}^0 = -p_1 x_{1j}$	$\Delta m_{1m}^0 = -p_1 x_{1m}$	$\sum_{j=1}^{m} \Delta m_{1j}^0 + \sum_{j=1}^{m} p_1 x_{1j} = 0$
2 $F_2; M_2^0 > \sum_{j=1}^{m} m_{2j}^0$	$\Delta m_{21}^0 = -p_2 x_{21}$	$\Delta m_{22}^0 = -p_2 x_{22}$	$\cdot\cdot\ \Delta m_{2j}^0 = -p_2 x_{2j}$	$\Delta m_{2m}^0 = -p_2 x_{2m}$	$\sum_{j=1}^{m} \Delta m_{2j}^0 + \sum_{j=1}^{m} p_2 x_{2j} = 0$
..i $F_i; M_i^0 > \sum_{j=1}^{m} m_{ij}^0$	$\cdot\cdot\ \Delta m_{i1}^0 = -p_i x_{i1}$	$\cdot\cdot\ \Delta m_{i2}^0 = -p_i x_{i2}$	$\cdot\cdot\ \Delta m_{ij}^0 = -p_i x_{ij}$	$\Delta m_{im}^0 = -p_i x_{im}$	$\sum_{j=1}^{m} \Delta m_{ij}^0 + \sum_{j=1}^{m} p_i x_{ij} = 0$
m $F_n; M_n^0 > \sum_{j=1}^{m} m_{nj}^0$	$\Delta m_{n1}^0 = -p_n x_{n1}$	$\Delta m_{n2}^0 = -p_n x_{n2}$	$\cdot\cdot\ \Delta m_{nj}^0 = -p_n x_{nj}$	$\Delta m_{nm}^0 = -p_n x_{nm}$	$\sum_{j=1}^{m} \Delta m_{nj}^o + \sum_{j=1}^{m} p_n x_{nj} = 0$
$\sum_{i=1}^{n} F_i = 0;$ $\sum_{i=1}^{n}\sum_{j=1}^{m} \Delta m_{ij}^o < \sum_{i=1}^{n} M_i^0$	$\sum_{i=1}^{n} \Delta m_{i1}^0 + \sum_{i=1}^{n} p_i x_{i1} = 0$	$\sum_{i=1}^{n} \Delta m_{i2}^0 + \sum_{i=1}^{n} p_i x_{i2} = 0$	$\sum_{i=1}^{n} \Delta m_{ij}^0 + \sum_{i=1}^{n} p_i x_{ij} = 0$	$\sum_{i=1}^{n} \Delta m_{im}^0 + \sum_{i=1}^{n} p_i x_{im} = 0$	

When an economic agent is buying goods or instruments the respective $\Delta m_{ij}^0 < 0$ and $p_i x_{ij} > 0$. If the agent is selling goods or instruments the signs change. The summation of these operations by columns and rows always equal zero. The equality $\sum_{i=1}^{n} F_i = 0$ reflects the Walras Law. The inequality

$\sum_{i=1}^{n} \sum_{j=1}^{m} \Delta m_{ij}^{0} < \sum_{i=1}^{n} M_i^0$ means that not all money is engaged in transactions so the precautionary motive of hoarding liquidity is effective.

The price of lending-borrowing is the interest rate, while all the other costs, related to monitoring and controlling, represent additional complement services, supplied by the banks. The role of interest rate is similar to Hirschleifer's optimal investment and lending-borrowing analysis (Hirschleifer: 1958: 329-52).

Equilibrium Convergence and Velocity of Circulation of Money

The equilibrium prices and quantities vectors presuppose some equilibrium real quantity of money under equilibrium conditions in the market economy. In particular, we assume $\hat{M}^0 = \hat{M}^0(\hat{p}, \hat{x})$, where $\hat{M}^0$ is the real equilibrium quantity of base money.

In our economy the monetary revenue may be generated only via exchanging goods or financial instruments for money. Only base money (current accounts and reserve accounts with the CB) can buy goods and financial instruments (quasi money).

We assume in addition that every unit circulation of money affects all markets[7].

We change the previous notation and assume not *n* goods and *m* agents, but *n* goods and *m* financial instruments. By unit circulation we mean that economic agents sell one unit of all *n+m* available goods and financial instruments. The units may be chosen randomly, so the circulation is any period that allows the economy to generate non-zero sales of all *n* goods and *m* financial instruments. The circulation period (t_c) equals the inverse of money velocity ($t_c = 1/v_0$).

We should make clear distinction between t_c and *T* or between the circulation and calculation periods. The period for which we calculate velocity (month, quarter, year and so on) is positively connected to velocity- the longer the period the more circulations can be performed. The circulation period (the time required for one circulation) equals the inverse of velocity.

We suppose that the economy needs several circulations to attain equilibrium. We can write:

(2) $$M_t^0 v_0 = P_t \cdot (\hat{p}_T \cdot \hat{x}_T) + \mu M_t^0 = M_t^0 (v_0 - \mu) = M_t^0 v_h = P_t \cdot (\hat{p}_t \cdot \hat{x}_t),$$

[7] In mathematical terms this may be guaranteed if we define velocity as kind of eigenvalue of respective matrix.

Where P_t is the price index of the real sector, and $\hat{p}_T \cdot \hat{x}_T$ is the equilibrium real income (output) for some fixed period T ($T > t_c = 1/v_o$) and μ is the money multiplier. It is evident from (2) that we separate the real and the financial sectors for analytical purposes. This is necessary in order to single out the interdependencies between both sectors, while in the same time they represent a cohesive general equilibrium system.

We need some additional qualifications in this respect. First of all, by separating the real and the financial (monetary) sectors, we introduced a new term in the right hand side of the Fisher equation of exchange. This is because the circulation of money not only generates income, but also creates financial instruments[8] (deposits and other instruments are sold on financial markets for money). The financial multiplier μ is some kind of "vertical" velocity of money.

The multiplier is interpreted as the number of circulations necessary to finance the total volume of issued instruments (market capitalization). The vertical velocity clearly depends on the average maturity of the existing financial instruments. The horizontal (real economy) velocity and the vertical (financial market) velocity must however be compatible, i.e. they should be calculated for the same period. For this purpose we assume that the calculation period T equals the average maturity of financial market instruments. We presume also that the average financial market instruments maturity exceeds or is equal to the circulation period of the horizontal velocity.

On the basis of (2) we can introduce several types of velocities.

First of all, the overall velocity v_o. The overall velocity equals the sum of "horizontal" and "vertical" velocities, or $v_0 = v_h + \mu : v_0, v_h, \mu \ni (0, +\infty)$.

Second, the "horizontal" or income velocity of money $v_h = v_0 - \mu$. This is the traditional concept of velocity from the Fisher equation.

Third "vertical" or financial velocity (multiplier) $\mu = v_0 - v_h$.

The above-mentioned three velocities are related to the money base. We can also introduce broad (quasi money) income velocity (taking into account that $M = \mu M^0$) or $v_b = \frac{v_0}{\mu} - 1$, where $\frac{v_0}{\mu} > 1$ by definition.

These four types of velocities may be characterized as follows.

The overall velocity is a transaction velocity in broad sense, i.e. including financial transactions. This velocity is supposed to increase in the long run. The

[8] According to the circuit theory of money the circulation of money generates new financial instruments and liquidates others, i.e. the existing broad quantity of money is a result of the primary monetary circulation and the multiplier process.

overall velocity never attains the maximum technical velocity since in this case all liquid reserves should shrink to the sum of the items in transmission of the balance sheets of economic agents. The transaction velocity may attain maximum if and only if the precautionary motive disappears.[9]

The “horizontal” or income velocity of money equals the difference between overall velocity and the multiplier. If we assume that in the long run the financial markets capitalization is growing faster then GDP in current prices, this velocity can be assumed to decline slowly. This is in conformity with the monetarist assumption, excluding the hypothesis of stability (the latter maybe assumed to be partially true about the overall velocity).

The “vertical” velocity is supposed to increase in the long run, but could be assumed to be highly unstable. The “vertical” velocity may also be interpreted to reflect the Keynesian finance motive of accumulation of money (Keynes, 1937) in the sense that additional quantity of money should circulate in order to make possible the exchange of financial instruments for money.

Finally, broad money income velocity dynamics conjecture is about slow decline in the long run plus some instability.

As already mentioned (1) is a variant of Fisher’s identity. Here the left side of (1) is interpreted as reflecting the inter-agent exchange decomposition while the right-hand part replicates market equilibrium properties.

If we divide (2) by the price index we obtain:

(3) $$\tilde{M}_t^0 v_t^h = \hat{p}_T \cdot \hat{x}_T ,$$

Where $\tilde{M}_t$ is the real money supply in moment *t*. Observe that we always have $\tilde{M}_t^0 < \hat{p}_T \cdot \hat{x}_T$ for $T > t_c = 1/v^h$, since otherwise we would have either $\tilde{M}_t^0 = \hat{p}_T \cdot \hat{x}_T$ or $\tilde{M}_t^0 > \hat{p}_T \cdot \hat{x}_T$. The equality would mean that all the income during period *T* is saved (financed by new lending).[10] Note also that $\tilde{M}_t^0 > \hat{p}_T \cdot \hat{x}_T$ is excluded by definition as far as we assume more then unit circulation. The inequality $\tilde{M}_t^0 < \hat{p}_T \cdot \hat{x}_T$ specifies that income velocity never equals unity, contrary to traditional CIA approach, for an appropriate choice of period *T*. Observe also that in (3) the financial sector is included implicitly since the horizontal velocity is the difference between overall velocity and the multiplier.

Now we are ready to formulate the following *Proposition*.

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[9]Consequently if we assume that in the long run the relative importance of the precautionary motive is increasing the overall velocity should decrease.

[10] Theories assuming unitary velocity of circulation reduce the economic analysis to periods where circulation and calculation periods coincide.

Proposition 1: If the real money supply is fixed at the optimal level,[11] and if at the initial point the real money income is below the equilibrium, that is if $p \cdot x < \hat{p} \cdot \hat{x}$ and $\hat{M}^{0}_{t_1} v^{h}_{t_1} < \hat{p} \cdot \hat{x}$, then an appropriate increase of income velocity, i.e. increase that does not exceed the velocity required at equilibrium, is necessary and sufficient condition for equilibrium convergence.

Proof:

Obviously the equilibrium income velocity is $\hat{v} = (\hat{p}_T \cdot \hat{x}_T) / \hat{M}^{0}_{t_1}; \hat{M}^{0}_{t_1} > 0$.

By definition $v^{h}_{t_1} < \hat{v}^{h}$. So only acceleration of velocity from $\hat{v}^{h}_{t_1}$ to $\hat{v}^{h}$ can guarantee equilibrium convergence and any convergence requires the above defined velocity increase.

The Proposition 1 establishes connection between income velocity and equilibrium convergence under specified conditions. Since the proposition requires fixed optimal real money supply it presupposes some kind of monetary policy. The Proposition is also quite restrictive since it implies that money (monetary base) market is in equilibrium while the other markets are not.

One important consequence of the Proposition 1 is that ceteris paribus deceleration of velocity indicates equilibrium divergence.

We can observe also that acceleration of the income velocity can take place via acceleration of horizontal velocity, via decline of multiplier or some combination of both. In the first case the economy adjusts by the real sector while the monetary is supposed to be in equilibrium. In the second case the adjustments takes place because some currency is transferred from the financial to the real sector increasing the output in the latter.

The relationship between money velocity and equilibrium is not new-fangled. According to Phelps (2007), an increase of money velocity is neutral if the economic agents have *correct* expectations, but if not and if the central bank is slow to respond, then any positive velocity shock would drive both the price level and money-wage level toward correspondingly higher paths.

In our context the process is reversed. If the economy is in equilibrium and if some autonomous increase (decrease) in demand or supply drives it out of balance, then we have overheating with physical production possibly above the optimum level. Overheating usually presupposes preceding velocity deceleration, reflecting excessive money creation. The latter situation is nevertheless below the social optimum in the sense that the resources are misallocated. Only velocity acceleration may re-establish the equilibrium even at the cost of higher inflation.

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[11] Fixing of the money supply on the optimum level does not necessarily mean equilibrium on the money market since the latter means optimal price (interest rate) and equilibrium on all other markets.

As we mentioned, our definition of equilibrium prices and quantities includes equilibrium real quantity of money as part of the integrated economy. By definition, the real quantity of money is less then the real equilibrium income for appropriate choice of periods and units. So the only free variable allowing for equilibrium convergence is the income velocity of money.

The dynamics of income velocity strongly depends on the monetary aggregate used. We used the concept of monetary base. We can however easily reformulate (3) in terms of broad money.

(4) $$\frac{\hat{M}_t}{\hat{\mu}_t}\hat{v}_t^h = \hat{p}_T \cdot \hat{x}_T ,$$

Where $\hat{\mu}_t$ is the (equilibrium) money multiplier and $\hat{M}_t$ is the real equilibrium broad money. Further we can define $\hat{v}_{t\,0}^b = \frac{\hat{v}_t^h}{\hat{\mu}_t}$, $\hat{v}_t^b$ being the (equilibrium) velocity of the broad money, so we obtain:

(5) $$\hat{M}_t \hat{v}_t^b = \hat{p}_T \cdot \hat{x}_T$$

In equation (5) the left hand side can also be expressed in terms of prices and quantities ($\hat{M}_t = \hat{p}_T^m \cdot x_T^m$) of the money (financial markets). So the broad money velocity denotes the proportion between the two markets for a given period. The equation (5) implies also that base money velocity is always higher than the broad money one. If the multiplier varies, broad and narrow money could have diverging dynamics.

The next problem is the stability of the equilibrium convergence. It can be reduced to the study of a system of *n+m* first order differential equations, specifying equilibrium convergence of respective markets. Here *n* stands for number of real sector markets and *m* for the number of financial markets.

We assume the excess demand per unit of money ($F_{1M} = F_1 / M_1^0, F_{2M} = F_2 / M_2^0, ... F_{(n+m)M} = F_{(n+m)} / M_{n+m}^0$; where F_{iM} is the excess demand on market i and M_i^0 is the quantity of money per market) on the respective market is a function velocities v_{ij},[12] i.e. $F_{iM} = F_{iM}[v_{i1}(t), v_{i2}(t)..v_{ij}(t)..v_{i,n+m}(t)]$, where v_{ij} is cross velocity and stands for the number of payments from market *j* to market

[12] The excess demand per unit of money is the excess demand velocity of money; consequently it is a function of cross velocities.

i per unit period. The excess demand is defined as $F_i = \hat{p}_i \hat{x}_i - p_i^t x_i^t$, i.e. as deviation from equilibrium.

If we assume that the base money is fixed at optimum level, then the excess demand per unit of money is nothing, but the difference between the optimum and effective velocity of money. The presence of financial intermediation allows for payments from any random market to any other market of the integrated real and financial economy.

In dynamical terms we can define the system as follows:

(6) $$\dot{F}(t) = \dot{F}(v_{ij}) = -V' F_m(t),$$

Where $F_m(t)$ is the excess demand per unit of money vector and $V' = [v_{ij}^{'}]$ is $n+m \times n+m$ matrix. The negative sign before V' reflects the assumption that the economic system is below the optimum and is converging to the equilibrium values of prices and quantities.

The solution of (6) is:

(7) $$\phi(t; F^0) = e^{V^{'}t} F^0,$$

Where, from the definition of the natural logarithm and the exponential function we have $e^{-V't} = -\sum_{k=0}^{\infty} V'^k \frac{t^k}{k!}$ (see Takayma, 1990); F^0 is an *n+m* vector of the initial excess demand.

The elements of the matrix $V^{'}$ are defined as $F_{iM}^{'} = \partial F_{iM} / \partial v_{ij} = \partial v_{ij} / \partial t = v_{ij}^{'}$ or they reveal the acceleration of income velocity of money from the j-*th* to the i-*th* market.[13]

In the process of equilibrium convergence the income velocity, if it exists, should meet the condition $v_{ij}^{'} \geq 0$.

Yet, we need some additional investigation into the matrixes V' and *V*, *V* being the matrix of velocities..

First of all, observe, that we can assume for *V* and its Jacobian matrix V', the realization of the following property:

(8) $$V \cdot m = \lambda m; V \cdot m - \lambda \,\mathrm{Im} = 0; (V - \lambda I)m = 0; \det(V - \lambda I) = 0.$$

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[13] We can replace markets by economic agents, as we shall do later, since the inter agents' payments are in the same time payments, related to specific markets with the same excess demand per monetary unit.

Here *V* is the matrix of velocities; *m* is the vector of quantities of money per market, λ is the eigenvalue of the matrix V, I is the identity matrix and $m \ni M^0$ is the monetary vector in circulation.[14] In order to apply (8), the matrix V should meet the usual requirements - to be nonnegative and indecomposable (Takayama, ibid).

The eigenvector *m* is unique up to a scalar multiple. It means that the inflation in equation (8) is indefinite. As it's well known, λ increases as any element of V increases. We interpret λ as velocity of money.[15] Consequently, the velocity depends on the dynamics of all markets or inter-agents transactions.

The equation (8) can be seen also as a formal derivation of the left hand side of Fisher's identity.

If the quantity of money in circulation is fixed and if we assume, without loss of generality, that $M^0 = 1$, then any increase of $V \cdot m$ requires intensification of the aggregate velocity. The same can be applied to the matrix V'. So we can rewrite (8) as:

(9) $$\phi(t; F^0) = P \cdot e^{-\Lambda' t} \cdot P^{-1} \cdot F^0 ,$$

Where P is a non-singular permutation matrix, Λ' is a diagonal matrix with all diagonal elements equal λ' and *t* is a column vector with all elements *t*. We assume $\lambda' = v'$. The negative sign before the acceleration of velocity guarantees the equilibrium convergence and its stability. We can assume further, that $F^0_{1M} = F^0_{2M} = ...F^{\ 0}_{(n+m)M}$, i.e. that the excess demand per unit of money is equal on all markets.

The latter assumption is based on arbitrage considerations- in a monetary economy with free disposability of money, the gains per unit of money (expressed as $\int_{v_0}^{\hat{v}} v_t^0 F_M(v) \delta v$), under equilibrium convergence, should be considered equal. The same conclusion is drawn by Kim (Kim, 2002) and named "Low of Equal

[14] The equation (8) signifies that the matrix V determines linear transformation from $m \ni M^0$ to the set of possible monetary income outcomes. If we replace V by V' we obtain mapping to the set of excess demand shifts.

[15] In the text we do not discuss details, but λ may be interpreted as "in motion" velocity and the respective quantities as amounts "in motion". Such (unobservable) velocity and quantities are optimum "disequilibrium transmission" values. In such a case lambda is the upper boundary of the observable velocity. The observable velocities reflect, in addition to transaction demand for money, also the precautionary, speculative and finance motives. The "money in motion" velocity is driven by micro economic forces, but its value and the quantity of money "in motion" are determined by systemic features, that is by the existence and length of closed circulation paths and by technological factors.

Marginal Velocities". It is also obvious that marginal velocities should converge to the prevailing interest rate level.

If we suppose in addition that v' *is* constant during convergence, then the time necessary to attain the equilibrium is $t_e = (\hat{v} - v_0)/v'$. So the equilibrium convergence, under our simplified assumptions, consists of immediate jump from $v' = 0$ at $t = t_0$, to velocity acceleration attaining some constant value of $v' = q$, up to moment $t_f = t_0 + t_e$, when the equilibrium is reached on all markets simultaneously and the velocity acceleration returns to initial point $v' = 0$.

Under the above conditions, any deviation from equilibrium may be assumed to generate convergence process, based on income velocity of money acceleration.

Note that our understanding of velocity differs from traditional view of income velocity of money and other velocities as some kind of average velocities. In our case the money velocity has systemic features, i.e. we define velocity as some kind of integrating variable. Non-zero velocity presupposes that all markets are interconnected[16] and some types of closed payments paths exist. This follows from the definition and the properties of matrix determinants.

The Interdependence between Real and Financial Subsystem under Equilibrium Convergence

We can also compare traditional general equilibrium market convergence to velocity convergence. Let's have a system of *n* real and *m* financial markets. The overall velocity is v_o. System convergence requires simultaneously rise in money velocity and negativity of real part eigenvalues of the conventional system's Jacobean.[17] The latter is defined in a traditional way- the excess demand is viewed as a function of all n+m real and financial market prices. The Jacobean itself is a function of the first partial derivatives of excess demand with respect to all *n+m* prices.

This system can be decomposed into four block matrices- A, B, C and D. A is $n \times n$ square matrix, representing partial derivatives of the n real market excess demand functions with respect to n real market prices. B is $n \times m$ matrix consisting of partial derivatives of the n real excess demand functions with respect to m financial market prices. Matrix B reflects the impact of financial over real sector. Matrix C is $m \times n$ matrix consisting of derivatives of financial markets excess demand functions with respect to real market price changes. Matrix C

[16] The assumption that all markets maybe interconnected via the circulation of money is not unrealistic if financial intermediation exists since banks can transfer liquidity between any pair of markets.

[17] The Jacobeans are transformed in accordance with the approach, developed in Appendix 1.

represents the feed back from real to financial sector. It is the feedback of the real sector to the financial sector. Finally matrix D is $m \times m$ square matrix including the first partial derivatives of financial excess demand functions with respect to financial instruments market prices.

Matrices A and D assume circulation of money in the real and the financial sectors respectively. Matrices B and C reflect transmission of liquidity from financial to real sector and from the real to financial sectors correspondingly by means of the financial intermediation. To put it differently matrices B and C consist of intersections between horizontal and vertical velocities.[18]

From the Leibniz formula we can deduce that total determinant equals the product of two expressions - $\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A)\det(D - CA^{-1}B)$. The block matrix equation indicates that in order to meet the positive eigenvalue constraint the first and second determinants should have the same signs.[19] If we interpret the first determinant as standing for real market self-convergence dynamics and the second determinant as the difference between financial market self-convergence properties minus the interaction impact, then we can draw some conclusions about sectors interdependences.

If self-convergence reaction of the real sector prevails ($\det(A) > 0$) then second determinant should be also positive ($\det(D - CA^{-1}B) > 0$). The latter is possible if the interaction impact is less than the self adjustment reaction of the financial sector. We can formulate this requirement also by putting the financial sector at first place.

By self convergence properties we mean that the partial derivative of every market excess demand function with respect to the same market price (see the Appendix). We assume in principle that the main diagonal elements of D are positive, but the financial markets are prone to instability in the long run (see the Appendix). The interdependencies among markets are taken into

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[18] For example, if market $i \ni n$ and market $j \ni m$ and money generated by selling financial instrument (bank obtains deposit) j are transferred to market i for purchase of investment goods, then money will increase both monetary income in sector i and financial assets in sector j (we assume for simplicity that deposit collection and lending occur simultaneously). This is intersection of horizontal and vertical circulation. If money, generated by selling products of sector $i \ni n$ are spent for inputs in sector $p \ni n$ we have horizontal circulation. In the same way if money received as loan are invested in other financial instruments (shares) we have vertical circulation.

[19] As it was mentioned earlier to guarantee the stability of the system of equations we need negative real parts of the roots of its characteristic equation of the main matrix. This is fulfilled if the main minors are positive.

account by the expression $CA^{-1}B$. Since this term is deduced from the financial market matrix, the interactions should be either negative or relatively small if the system is to converge. Negativity of interdependence term in principle corresponds to gross substitutability hypothesis, which is rooted in the idea for the possible total equilibrium. So in a monetary economy the substitutability is realized through the intersection of horizontal and vertical velocity. If these channels are not functioning appropriately the system maybe destabilized.

On the basis of this analysis we can define the following three main types of system convergences- monetarist, Neo-Keynesian and real business cycles.

The monetarist type of convergence maybe understood as process implying stable overall velocity of money ($v_0^{'} = 0$). Since $v^0 = v^h + \mu$ the overall stability is compatible with opposite signs of horizontal and vertical velocities dynamics. Broad money velocity $v^b = \frac{v^h}{\mu}$ may have any sign and be both stable and unstable. On the other hand, the fixity of the overall velocity implies $\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = 0$.[20] Since we cannot assume the real and financial sectors to be non-converging under external shocks, we should put forward the condition $\det(D - CA^{-1}D) = 0$.

The interpretation is that under fixed velocity, the interaction element compensates for the monetary sector self-adjustment reaction. Consequently money is neutral in the sense that financial sector adjustment simply compensates for interactions, while the real one converges to equilibrium. This is possible if gross substitutability is effective. The opposite situation may occur when, for example, the movements of the oil price are weakly synchronized with financial markets quotations[21]. So the monetarism is about coordinated long term expectations about real and financial economy.

The Keynesian convergence may be interpreted in two different ways. First, as an economic system evolution where financial markets adjust immediately, but the real sector is lagging behind because of different types of price and wage rigidities (Rogoff, 2002). In our context this means we can assume $D = 0$, so we have $\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A)\det\left(-CA^{-1}B\right) > 0$. The reading is that any possible

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[20] Zeros acceleration means in the same time zero determinant of the excess demand Jacobean.

[21] Under the gross substitutability assumption the increase of the price of the energy should be positively correlated with stock exchange quotations. On the contrary, the sharp rise of oil prices simultaneously with the decline of stock prices was one of the symptoms of the financial crisis in 2008.

real sector convergence ($\det(A) > 0$) must be accompanied by negative interaction term since the financial sector is stationary (adjustment has already taken place). This indicates also that the financial sector overreacts (overshoots) in the sense that it takes ex ante into account the impact of the ex post real sector correction. In terms of velocities the above conjecture implies fixed multiplier and (eventually) accelerating horizontal velocity.

The Neo-Keynesian situation is just the opposite of the monetarist one. The gross substitutability holds, but the financial sector self adjustment signals cannot be transmitted to the real one. The government policy of deficit financing is nothing but hastened transmission of liquidity from financial to real sector in order to compensate for constrained demand in latter. Such a policy can be beneficial if it does not disturb the already established financial market price structure. The former is possible if excess bank reserves prevail or if fiscal and monetary policies are appropriately coordinated.

The alternative Keynesian interpretation implies economic destabilization initiated by the financial sector. This means that the main diagonal elements of D are negative. Such a situation reflects puffing up and bursting of financial bubble. The second variant requires even stronger anti cyclical policy measures. This can be associated with the present global financial crisis.

The third type is the real business cycle type adjustment. The theory of the real business cycles, at least its strong form, assumes that the real sector shocks cause monetary sector shifts but not vice-versa (McCalum&Benett, 1989). The overall velocity should be determined by the real sector needs. In our context this means that $B = 0$. In such a case we have $\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A)\det(D)$. The synchronized convergence of the real and financial sectors imposes positive determinants of A and D and total alienation of real from financial markets, since, according to our model, the lack of feed back from monetary to real sector cuts also the effects of real over the financial one, what is not viable. In terms of velocities this means $\partial v^h / \partial \mu = \partial \mu / \partial v^h = 0$.

Nevertheless the real business cycle theory correctly stresses on the self-adjustment properties of both financial and real sectors in case of unimportant interaction term. Weak real business cycle adjustment may be the most realistic type of adjustment and even be compatible with the post Keynesian approach, if we assume not disturbing government financing and exclude financial bubbles.

We need also to precise that we do not constrain the factors that can disturb the economy and induce convergence. These can by technological shocks, energy and row material prices shocks, fiscal and monetary policy shocks and so on (see McGrattan, 2006).

On the other hand in our case there is a general regularity concerning the financial markets. To guarantee the stability of the integrated system of financial and real markets we need the fulfillment of the gross substitutability principle what reduces to negative connection between the excess demand on all markets and the prices of the respective goods. In the case of financial markets however the price increase may generate not decrease but increase in demand. Such behavior is characteristic under speculative financial bubbles. That's why we can expect that the financial system is behind the economic crises.

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Appendix

The exposition is based on Takayama (Takayma, 1990). We assume linear approximation system of n+m equations. The system can be defined as:

(1) $$\frac{dq(t)}{dt} = A \cdot q(t)$$

Where $q_i(t) = p_i(t) - \hat{p}_i, i = 1,2....,n+m$; $\hat{p}_i$ is the equilibrium price.

$A = [a_{ij}]$ where the elements a_{ij} equal $\partial f_i / \partial p_j, i, j = 1,2,...n+m$, evaluated at $p = \hat{p}$. The $f_i[p(t)]$ are the excess demand functions and $\frac{dp_i(t)}{dt} = f_i[p(t)], i = 1,2,...n+m$.

The equation (1) converges, that is $q(t) \to 0$ as $t \to \infty$ and $p(t) \to \hat{p}$ if and only if the real parts of the eigenvalues of A are negative.

In principle the above condition implies complicated proofs and additional assumptions about economic agents' behavior (Takayama, 1990). We will not follow this approach.

In spite, we will use one of the properties of matrix determinants, namely the equation $\det(A) = e^{tr(\ln A)}$. Clearly, the logarithms of the main diagonal elements of the matrix A (more precisely of all principal minors of A) may exist if and only if all the elements $a_{ii} > 0$. If this requirement is fulfilled, the matrix A will have all negative real parts of the roots of its characteristic equation.

In principle however we should expect elements a_{ii} to be negative and the non-diagonal elements to be positive. This follows from the lows of supply and demand and from the gross-substitutability hypothesis. To get the necessary result, observe that we can multiply equation (1) by -1. Thus we obtain $-\frac{dq(t)}{dt} = -A \cdot q(t)$. By introducing new variables $\frac{d\tilde{q}(t)}{dt} = -\frac{dq(t)}{dt} = \hat{p} - p(t)$ and $\tilde{A} = -A$ we get the solution of (1):

(2) $$q(t) = e^{\tilde{A}t} \cdot q(t_0) = P \cdot e^{\Lambda t} \cdot P^{-1} \cdot q(t_0),$$

Where Λ is a matrix with main diagonal elements consisting of eigenvalues of $\tilde{A}$ and P is non singular matrix. From equation (2) it is apparent that negative real parts guarantee the convergence $q(t) \to 0$ as far as $t \to \infty$.

The non-negativity of the main diagonal elements of matrix $\tilde{A}$ has clear economic interpretation. It simply means that the laws of supply and demand in terms of partial equilibrium hold. General equilibrium conditions, under augmented Clower constraint, require that every market excess demand variations are negatively correlated with the same market price changes, but generate on other markets excess demand shifts, positively linked to cross price movements (gross substitutability). More precisely, if the liquid excess demand on some market declines, it is either transferred to money market or to other financial or real markets, i.e. gross substitutability has factual dimension. In a monetary economy, our equilibrium convergence condition and the standard gross substitutability conditions are equivalent.

The inverse relationship between market price and excess demand seems plausible in the short run (fixed production capacities). In the long run, especially on financial markets with imperfect information, quotations going up are seldom interpreted by economic agents as encouraging signals boosting up demand. Thus excess demand will increase along with price escalation. Bursting financial bubbles may trigger over intensive reverse process. Therefore we may expect the financial sector to initiate economic instability.

2.06.2010